

Variational iteration method: New development and applications

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Abstract

Variational iteration method has been favourably applied to various kinds of nonlinear problems. The main property of the method is in its flexibility and ability to solve nonlinear equations accurately and conveniently. In this paper recent trends and developments in the use of the method are reviewed. Major applications to nonlinear wave equation, nonlinear fractional differential equations, nonlinear oscillations and nonlinear problems arising in various engineering applications are surveyed. The confluence of modern mathematics and symbol computation has posed a challenge to developing technologies capable of handling strongly nonlinear equations which cannot be successfully dealt with by classical methods. Variational iteration method is uniquely qualified to address this challenge. The flexibility and adaptation provided by the method have made the method a strong candidate for approximate analytical solutions.

This paper outlines the basic conceptual framework of variational iteration technique with application to nonlinear problems. Both achievements and limitations are discussed with direct reference to approximate solutions for nonlinear equations. A new iteration formulation is suggested to overcome the shortcoming. A very useful formulation for determining approximately the period of a nonlinear oscillator is suggested. Examples are given to illustrate the solution procedure.

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1. Introduction

Very recently it was recognized that the variational iteration method [1–9] can be an effective procedure for solution of various nonlinear problems without usual restrictive assumptions. The method, extensively worked out by numerous authors, has been maturing into a fully fledged theory, more and more merits have been discovered and some modifications are suggested to overcome the demerit arising in the solution procedure. Applications of the method have been enlarged due to its flexibility, convenience and accuracy. A guided tour through the mathematics needed for a proper understanding of variational iteration method as applied to various nonlinear problems is available on Ref. [9], for a relatively comprehensive survey on the method and its applications, the reader is referred to a review article [5] or a monograph [8].

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2. Applications of the variational iteration method

Recently, some rather extraordinary virtues of the method have been exploited, and wide applications have been found in various fields. Generally one iteration leads to highly accurate solution, in other cases, we can obtain a series which converges fast to the exact solution.

2.1. Fractional differential equations

Fractional derivative has received considerable interest in recent years though it is a very old topic. In many applications, fractional derivatives provide with more accurate models of systems under consideration. Although the subject of the fractional differential equations is very old, it does not appear to have been considered systematically in modern literature, and its applicability range is severely limited, due possibly to the difficulty of solving the problems with fractional derivatives even with numerical simulation. So, most problems are reduced to partial differential equations, leading to loss of most significant information. Now the condition becomes better and better, the variational iteration method is a strong candidate for approximate analytical solutions of both linear and nonlinear fractional differential equations. Ji-Huan He first applied the variational iteration method to fractional differential equations [2], revealing a great success. Most fractional differential equations do not have exact analytic solutions, so approximation by variational iteration method is much needed. For example, Abbasbandy applied the variational iteration method to Riemann–Liouville’s fractional derivatives [10], Draganescu and his colleagues to nonlinear vibration with fractional damping [11], Momani and his colleagues applied the method to fluid mechanics where the fractional derivative was successfully applied [12–15].

2.2. Wave solutions

The variational iteration method is an effective method for searching for various wave solutions including periodic solutions, solitons, compacton solutions without linearization or weak nonlinearity assumptions [16–28]. The main feature of the method is that the initial solution can be chosen with some unknown parameters in the form of the searched solutions. For example, for periodic solution we can begin with an initial solution in the form

$$u_0(\xi) = a \sin(\omega\xi + \theta_0), \quad \xi = x + ct \quad (1)$$

where ω is an unknown parameter to be further determined after one or few iterations.

For solitons, we begin with

$$u_0 = p \operatorname{sech}^2(q\xi), \quad \xi = x + ct \quad (2)$$

where p and q are the unknown parameters to be further identified after one or few iterations. For a more general form for solitary solutions we assume that

$$u_0(\xi) = \frac{\sum_{i=-n}^m c_i e^{i\xi}}{\sum_{i=-p}^q b_i e^{i\xi}}, \quad \xi = x + ct \quad (3)$$

where c_i and b_i are constants to be further determined.

For discontinuous solitons, we can assume, for example, in the following form:

$$u(\xi) = p \exp(-q|\xi|), \quad \xi = x + ct \quad (4)$$

where p and q are the unknown parameters to be further identified.

For compacton-like solution, we assume the solution has the form [29–31]

$$u_0(x, t) = \frac{a \sin^2(kx + wt)}{b + c \sin^2(kx + wt)}, \quad (5)$$

where a , b , k , and w are unknown constants further to be determined.

In order to identify the constants in the initial solution, we can incorporate the initial/boundary conditions after one or few iterations. If more unknown parameters in the initial solution, we can use the following relationship [7]:

$$u_n(x, t) = u_{n+1}(x, t), \tag{6}$$

and

$$\frac{\partial^k}{\partial t^k} u_n(x, t) = \frac{\partial^k}{\partial t^k} u_{n+1}(x, t). \tag{7}$$

2.3. Applications to various nonlinear problems arising in engineering

The variational iteration method has been shown to solve effectively, easily, and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions. He applied his method to autonomous ordinary differential systems [4] and nonlinear equations with convolution product nonlinearity [3], Abulwafa et al. to nonlinear coagulation problem with mass loss [32] and to nonlinear fluid flows in pipe-like domain [33], Ariel et al. to axisymmetrical flow over a stretching sheet [34], D’Acunto [35,36], Xu [37], Draganescu [38], Ganji and Sadighi to nonlinear heat transfer [39], Marinca [40] and He [1] to nonlinear oscillators, Draganescu and Capalnasan to nonlinear relaxation phenomena in polycrystalline solids [38], Lu to two-point boundary value problems [41], Sweilam et al. to nonlinear thermoelasticity [42–44], Tatari and Dehghan to a semi-linear inverse parabolic equation [45], Liu to ion acoustic plasma wave [46] and nonlinear oscillators with discontinuities [47], Siddiqui et al. to non-Newtonian flows [48]. Most authors found that the shortcomings arising in the Adomian method can be completely eliminated by the variational iteration method.

2.4. Convergence

Generally one iteration leads to high accurate solution by variational iteration method if the initial solution is carefully chosen with some unknown parameters. If we begin with $u_0(x, t) = u(x, 0)$, an series solution can be obtained. The convergence of the method is systematically discussed by Tatari and Dehghan [49], comparison of the method with Adomian method was conducted by many authors via illustrative examples, especially Wazwaz gave a completely comparison between the two methods [50], revealing the variational iteration method has many merits over the Adomian method; it can completely overcome the difficulty arising in the calculation of the Adomian polynomial.

Though the variational iteration method leads to fast convergent solutions, unnecessary calculation arises in the solution procedure. To illustrate the demerit, we consider a simple example.

$$u_t + u^2 = 0 \quad u(0) = 1. \tag{8}$$

In view of the variational iteration method, we construct the following iteration formulation:

$$u_{n+1} = u_n - \int_0^t (u'_n(s) + u_n^2(s)) ds. \tag{9}$$

If we begin with $u_0(t) = u(0) = 1$, we can obtain a convergent series:

$$\begin{aligned} u_0(t) &= 1 \\ u_1 &= 1 - t \\ u_2 &= 1 - t + t^2 - \frac{1}{3}t^3. \end{aligned}$$

The last term in u_3 is unnecessary, so in the solution procedure we suggest the following iterations, which converge to the exact solution $u(t) = 1/(1 + x)$:

$$\begin{aligned} u_0(t) &= 1 + O(t) \\ u_1 &= 1 - t + O(t^2) \\ u_2 &= 1 - t + t^2 + O(t^3) \end{aligned}$$

$$\begin{aligned}u_3 &= 1 - t + t^2 - t^3 + O(t^4) \\u_4 &= 1 - t + t^2 - t^3 + t^4 + O(t^5).\end{aligned}$$

In order to accelerate the convergent rate, various other modifications were suggested, for example variational iteration–Pade method [51], variational iteration–Adomian method [16], variational iteration–differential transform method [52].

Hereby we provide another way of accelerating the series solutions. Differentiating both sides of Eq. (8) with respect to t results in

$$u'' + 2uu' = 0. \quad (10)$$

In view of Eq. (8), the first-order differential equation is turned out to be a second-order differential equation, which reads

$$u'' - 2u^3 = 0, \quad u(0) = 1, \quad u'(0) = -1. \quad (11)$$

The iteration formation can be readily obtained, which reads

$$u_{n+1}(t) = u_n(t) + \int_0^t (s-t)(u_n'' - 2u_n^3) ds. \quad (12)$$

Beginning with $u_0(t) = u(0) + u'(0)t = 1 - t$, by the iteration formulation (12) we have

$$u_1(t) = 1 - t - 2 \int_0^t (s-t)(1-s)^3 ds = 1 - t + t^2 - t^3 + O(t^4). \quad (13)$$

It is same as that needed for four iterations by the iteration formulation (9).

Abassy, El-Tawil and Zoheiry suggested another effective modification [51,53]. Consider a general nonlinear equation

$$Lu(x, t) + Ru(x, t) + Nu(x, t) = 0, \quad u(x, 0) = f(x), \quad (14)$$

where $L = \frac{\partial}{\partial t}$, R is a linear operator, which has partial derivatives with respect to x and $Nu(x, t)$ is a nonlinear term. Abassy et al. constructed an iteration formulation in the form [51,53]:

$$U_{n+1} = U_n - \int_0^t \{R(U_n - U_{n-1}) + (G_n - G_{n-1})\} d\tau, \quad (15)$$

where $U_{-1} = 0$, $U_0 = f(x)$ and $G_n(x, t)$ is calculated from the relation

$$NU_n(x, t) = G_n(x, t) + O(t^{n+1}). \quad (16)$$

An approximate solution can be obtain approximately in the form

$$u(x, t) \cong U_n(x, t) \quad (17)$$

where n is the final iteration step.

3. Development of variational iteration method

The variational iteration method [1–9] has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions. We summarize some useful iteration formulae:

$$(I) \begin{cases} u' + f(u, u') = 0 \\ u_{n+1}(t) = u_n(t) - \int_0^t \{u_n'(s) + f(u_n, u_n')\} ds \end{cases} \quad (18a)$$

$$(II) \begin{cases} u' + \alpha u + f(u, u') = 0 \\ u_{n+1}(t) = u_n(t) - \int_0^t e^{\alpha(s-t)} \{u'_n(s) + \alpha u_n(s) + f(u_n, u'_n)\} ds \end{cases} \quad (19a)$$

$$(III) \begin{cases} u'' + f(u, u', u'') = 0 \\ u_{n+1}(t) = u_n(t) + \int_0^t (s-t) \{u''_n(s) + f(u_n, u'_n, u''_n)\} ds \end{cases} \quad (20a)$$

$$(IV) \begin{cases} u'' + \omega^2 u + f(u, u', u'') = 0 \\ u_{n+1}(t) = u_n(t) + \frac{1}{\omega} \int_0^t \sin \omega(s-t) \{u''_n(s) + \omega^2 u_n(s) + f(u_n, u'_n, u''_n)\} ds \end{cases} \quad (21a)$$

$$(V) \begin{cases} u'' - \alpha^2 u + f(u, u', u'') = 0 \\ u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{2\alpha} (e^{\alpha(s-t)} - e^{\alpha(t-s)}) \{u''_n(s) - \alpha^2 u_n(s) + f(u_n, u'_n, u''_n)\} ds \end{cases} \quad (22a)$$

$$(VI) \begin{cases} u''' + f(u, u', u'', u''') = 0 \\ u_{n+1}(t) = u_n(t) - \int_0^t \frac{1}{2} (s-t)^2 \{u'''_n(s) + f(u_n, u'_n, u''_n, u'''_n)\} ds \end{cases} \quad (23a)$$

$$(VII) \begin{cases} u^{(4)} + f(u, u', u'', u''', u^{(4)}) = 0 \\ u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{6} (s-t)^3 \{u^{(4)}_n(s) + f(u_n, u'_n, u''_n, u'''_n, u^{(4)}_n)\} ds \end{cases} \quad (24a)$$

$$(VIII) \begin{cases} u^{(n)} + f(u, u', u'', \dots, u^{(n)}) = 0 \\ u_{n+1}(t) = u_n(t) + (-1)^n \int_0^t \frac{1}{(n-1)!} (s-t)^{n-1} \{u^{(n)}_n(s) + f(u_n, u'_n, u''_n, \dots, u^{(n)}_n)\} ds. \end{cases} \quad (25a)$$

If we write

$$(I) v(t) = - \int_0^t f ds \quad (18b)$$

$$(II) v(t) = - \int_0^t e^{\alpha(s-t)} f ds \quad (19b)$$

$$(III) v(t) = \int_0^t (s-t) f ds \quad (20b)$$

$$(IV) v(t) = \frac{1}{\omega} \int_0^t \sin \omega(s-t) f ds \quad (21b)$$

$$(V) v(t) = \int_0^t \frac{1}{2\alpha} (e^{\alpha(s-t)} - e^{\alpha(t-s)}) f ds \quad (22b)$$

$$(VI) v(t) = - \int_0^t \frac{1}{2} (s-t)^2 f ds \quad (23b)$$

$$(VII) v(t) = \int_0^t \frac{1}{6} (s-t)^3 f ds \quad (24b)$$

$$(VIII) v(t) = \int_0^t \frac{1}{(n-1)!} (s-t)^{n-1} f ds \quad (25b)$$

by a simple operation we have

$$(I) v' + f = 0 \quad (18c)$$

$$(II) v' + \alpha v + f = 0 \quad (19c)$$

$$(III) v'' + f = 0 \quad (20c)$$

$$(IV) v'' + \omega^2 v + f = 0 \quad (21c)$$

$$(V) v'' - \alpha^2 v + f = 0 \quad (22c)$$

$$(VI) v''' + f = 0 \quad (23c)$$

$$(VII) v^{(4)} + f = 0 \quad (24c)$$

$$(VIII) v^{(n)} + f = 0. \quad (25c)$$

Consider Eq. (21b), differentiating both sides of Eq. (21b) with respect to t results in

$$\begin{aligned} v'(t) &= \frac{1}{\omega} [f(u(s), u''(s), u''(s)) \sin \omega(s-t)]_{s=t} - \int_0^t f(u(s), u''(s), u''(s)) \cos \omega(s-t) ds \\ &= - \int_0^t f(u(s), u''(s), u''(s)) \cos \omega(s-t) ds \end{aligned} \quad (26)$$

and

$$\begin{aligned} v''(t) &= -[f(u(s), u''(s), u''(s)) \cos \omega(s-t)]_{s=t} - \omega \int_0^t f(u(s), u''(s), u''(s)) \sin \omega(s-t) ds \\ &= -f(u, u', u'') - \omega \int_0^t f(u(s), u''(s), u''(s)) \sin \omega(s-t) ds. \end{aligned} \quad (27)$$

In view of Eq. (21b) we have

$$v''(t) = -f(u, u', u'') - \omega^2 v, \quad (28)$$

this is equivalent to Eq. (21c). That means v is a special solution. Accordingly we can construct a very simple iteration formulation in the form

$$u_{n+1}(t) = u_0(t) + v_n(t), \quad (29)$$

where u_0 is initial solution with or without unknown parameters. In case of no unknown parameters, u_0 should satisfy initial/ boundary conditions. When some unknown parameters are involved in u_0 , the unknown parameters can be identified by initial/boundary conditions after few iterations, this technology is very effective in dealing with boundary problems, see Examples 2 and 3 below.

For Eq. (20a), we can begin with

$$u_0(t) = u(0) \cos \omega t + \frac{1}{\omega} u'(0) \sin \omega t \quad (30)$$

and the iteration is reduced to a simple formulation, which reads

$$u_{n+1}(t) = u(0) \cos \omega t + \frac{1}{\omega} u'(0) \sin \omega t + \frac{1}{\omega} \int_0^t f(u_n, u'_n, u''_n) \sin \omega(s-t) ds. \quad (31)$$

Now all of the iteration formulae (18a)–(25a) can be re-written in the following simple formulae:

$$(I) \begin{cases} u' + f(u, u') = 0 \\ u_{n+1}(t) = u_0(t) - \int_0^t f(u_n, u'_n) ds \end{cases} \quad (18d)$$

$$(II) \begin{cases} u' + \alpha u + f(u, u') = 0 \\ u_{n+1}(t) = u_0(t) - \int_0^t e^{\alpha(s-t)} f(u_n, u'_n) ds \end{cases} \quad (19d)$$

$$(III) \begin{cases} u'' + f(u, u', u'') = 0 \\ u_{n+1}(t) = u_0(t) + \int_0^t (s-t) f(u_n, u'_n, u''_n) ds \end{cases} \quad (20d)$$

$$(IV) \begin{cases} u'' + \omega^2 u + f(u, u', u'') = 0 \\ u_{n+1}(t) = u_0(t) + \frac{1}{\omega} \int_0^t \sin \omega(s-t) f(u_n, u'_n, u''_n) ds \end{cases} \quad (21d)$$

$$(V) \begin{cases} u'' - \alpha^2 u + f(u, u', u'') = 0 \\ u_{n+1}(t) = u_0(t) + \int_0^t \frac{1}{2\alpha} (e^{\alpha(s-t)} - e^{-\alpha(s-t)}) f(u_n, u'_n, u''_n) ds \end{cases} \quad (22d)$$

$$(VI) \begin{cases} u''' + f(u, u', u'', u''') = 0 \\ u_{n+1}(t) = u_0(t) - \int_0^t \frac{1}{2} (s-t)^2 f(u_n, u'_n, u''_n, u'''_n) ds \end{cases} \quad (23d)$$

$$(VII) \begin{cases} u^{(4)} + f(u, u', u'', u''', u^{(4)}) = 0 \\ u_{n+1}(t) = u_0(t) + \int_0^t \frac{1}{6} (s-t)^3 f(u_n, u'_n, u''_n, u'''_n, u^{(4)}_n) ds \end{cases} \quad (24d)$$

$$(VIII) \begin{cases} u^{(n)} + f(u, u', u'', \dots, u^{(n)}) = 0 \\ u_{n+1}(t) = u_0(t) + (-1)^n \int_0^t \frac{1}{(n-1)!} (s-t)^{n-1} f(u_n, u'_n, u''_n, \dots, u^{(n)}_n) ds. \end{cases} \quad (25d)$$

Example 1. We consider a linear fourth-order intergro-differential equation [54]

$$\begin{aligned} y^{(4)}(x) &= x(1 + e^x) + 3e^x + y(x) - \int_0^x y(x) dx, & y(0) &= 1, & y(1) &= 1 + e, \\ y''(0) &= 2, & y''(1) &= 3e. \end{aligned} \quad (32)$$

We introduce $f(x)$ as

$$f(x) = \int_0^x \int_0^\xi \left[s(1 + e^s) + 3e^s + y(s) - \int_0^s y(s) ds \right] d\xi dx. \quad (33)$$

We begin with

$$y_0(x) = 1 + e^x (a + bx + cx^2 + dx^3) \quad (34)$$

where $a, b, c,$ and d are unknown constants to be further determined.

By the iteration formulation (20d), we have

$$\begin{aligned} y_1(x) &= y_0(x) + \int_0^x (s-x) f(s) ds \\ &= [dx^3 + a + 1 + 10c - 90d - b + (-3d + c)x^2 + (-1 + 30d - 2c + b)x] e^x \\ &\quad + (-8c + b + 60d)x + 90d + \left(-\frac{1}{12}c - \frac{1}{24} + \frac{1}{4}d + \frac{1}{24}b - \frac{1}{24}a \right) x^4 \\ &\quad + \left(\frac{1}{6}b + \frac{1}{3} - \frac{2}{3}c + 3d \right) x^3 + b + \left(-3c + \frac{1}{2}b + \frac{1}{2} + 18d \right) x^2 - 10c. \end{aligned} \quad (35)$$

If the first-order approximate solution is enough, by the initial /boundary conditions $y(0) = 1, y(1) = 1+e, y''(0) = 2, y''(1) = 3e,$ we can identify the unknown constants as follows:

$$a = 0, \quad b = 0.9704, \quad c = 0.0296, \quad d = 0.0803. \quad (36)$$

So we obtain the following first-order approximate solution

$$\begin{aligned} y &= (0.0803x^3 + 2.3202x - 0.2113x^2 - 6.9014)e^x \\ &\quad + 0.0164x^4 + 0.7162x^3 + 10.3418x^2 + 7.9014 + 5.5516x. \end{aligned} \quad (37)$$

Comparison of the approximate solution, Eq. (49) with the exact solution $y = 1 + xe^x$ is illustrated in Fig. 1.

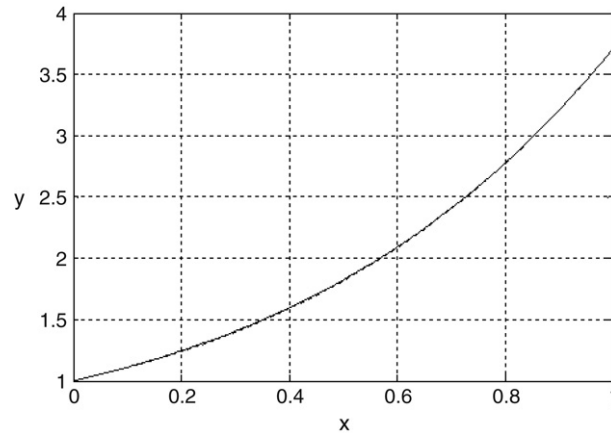


Fig. 1. Comparison of the approximate solution obtained by variational iteration method with the exact solution. Dashed line: approximate solution, Eq. (37); continuous line: exact solution.

Example 2. Consider another nonlinear fourth-order BVP [54]:

$$y^{(4)}(x) = 1 + \int_0^x e^{-x} y^2(x) dx, \quad 0 < x < 1, \quad y(0) = 1, \quad y(1) = e, \quad y''(0) = 1, \quad y''(1) = e. \quad (38)$$

We use the iteration formulation (20d) with

$$f(x) = \int_0^x \int_0^\xi \left[1 + \int_0^s e^{-s} y^2(s) ds \right] d\xi dx. \quad (39)$$

We assume that the initial solution can be written in the form

$$y_0(x) = 1 + e^x (a + bx + cx^2 + dx^3) \quad (40)$$

where $a, b, c,$ and d are unknown constants to be further identified.

Using (20d), we have

$$\begin{aligned} y_1(x) = & [-d^2 x^6 + (-2cd + 30d^2)x^5 + (50cd - c^2 - 2bd - 450d^2)x^4 \\ & + (d - 600cd + 4200d^2 + 20c^2 - 2bc - 2ad + 40bd)x^3 \\ & + (c + 30bc - 360bd - 180c^2 - b^2 + 4200cd + 30ad - 25200d^2 - 2ac)x^2 \\ & + (b - 180bc + 10b^2 - 2ab + 20ac - 16800cd + 1680bd + 840c^2 + 90720d^2 - 180ad)x \\ & + a + 10ab - 30b^2 - 3360bd + 30240cd - 1680c^2 + 420ad - a^2 - 60ac + 420bc - 151200d^2]e^x \\ & - \frac{d}{3360}x^8 - \frac{c}{1260}x^7 - \frac{b}{360}x^6 - \frac{a}{60}x^5 - 420ad + 151200d^2 + 60ac + 30b^2 + a^2 + 3360bd \\ & - 30240cd - 420bc + 1680c^2 - 10ab \\ & + \left(-10cd + \frac{a^2}{24} + 2bd - \frac{ab}{12} - \frac{1}{12} + \frac{b^2}{12} + 30d^2 - \frac{ad}{2} - \frac{bc}{2} + c^2 + \frac{ac}{6} \right) x^4 \\ & + \left(\frac{1}{6} + \frac{a^2}{6} + b^2 - 8ad + 840d^2 - 240cd + 40bd - 8bc - \frac{2}{3}ab + 2ac + 20c^2 \right) x^3 \\ & + \left(180c^2 - 2520cd - 60ad + 360bd + 10080d^2 - 3ab + 6b^2 - 60bc + 12ac - \frac{1}{2} + \frac{a^2}{2} \right) x^2 \\ & + (20b^2 - 13440cd + 1680bd + 60480d^2 - 240bc + 40ac - 240ad + 1 + a^2 + 840c^2 - 8ab)x + e^{-x}. \quad (41) \end{aligned}$$

Using the initial/boundary conditions $y(0) = 1, y(1) = e, y''(0) = 1, y''(1) = e$ to determine the unknown constants in Eq. (41):

$$a = 0, \quad b = 1.0499, \quad c = 0.5499, \quad d = 0.15117. \quad (42)$$

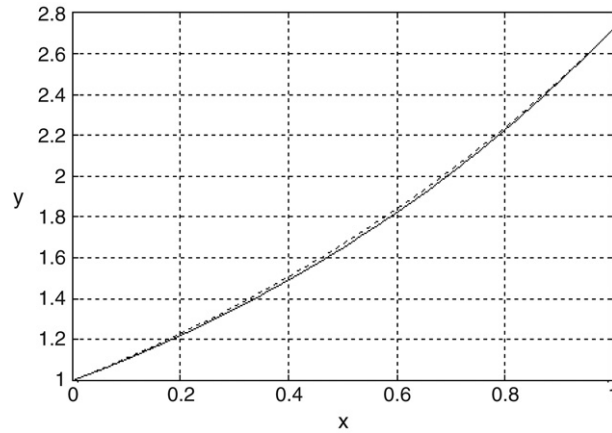


Fig. 2. Comparison of the approximate solution obtained by variational iteration method with the exact solution. Dashed line: approximate solution, Eq. (43), continuous line: exact solution.

So we obtain the needed first-order approximate solution:

$$\begin{aligned}
 y(x) = & (-15.0602x^4 + 0.8518x^5 - 1055.6x^2 + 159.5627x^3 - 0.0229x^6 + 4106.4x - 7286.1)e^x \\
 & - 4.4992 \times 10^{-5}x^8 + 4.3643 \times 10^{-4}x^7 - 7286.1 + 2.4339x^4 + 57.6318x^3 \\
 & + 592.1673x^2 + 3181.7x + e^{-x}.
 \end{aligned}
 \tag{43}$$

Comparison of the approximate solution, Eq. (43) with the exact solution $y = e^x$ is illustrated in Fig. 2.

Example 3. We consider a nonlinear oscillator [5,6]

$$u'' + \varepsilon u^3 = 0, \quad u(0) = A, \quad u'(0) = 0.
 \tag{44}$$

In our study, the parameter ε does not require to be small, $0 < \varepsilon < \infty$. We rewrite Eq. (44) in the form

$$u'' + \omega^2 u + f = 0, \quad u(0) = A, \quad u'(0) = 0,
 \tag{45}$$

where $f = \varepsilon u^3 - \omega^2 u$.

Its iteration formulation can be easily obtained

$$u_{n+1}(t) = u_0(t) + \frac{1}{\omega} \int_0^t \sin \omega(s-t)(\varepsilon u_n^3 - \omega^2 u_n) ds.
 \tag{46}$$

Beginning with

$$u_0 = A \cos \omega t.
 \tag{47}$$

we have

$$\begin{aligned}
 u_1(t) &= A \cos \omega t + \frac{1}{\omega} \int_0^t \sin \omega(s-t)(\varepsilon A^3 \cos^3 s - \omega^2 A \cos s) ds \\
 &= A \cos \omega t + \frac{1}{\omega} \int_0^t \sin \omega(s-t) \left(\frac{1}{4} \varepsilon A^3 \cos 3\omega s + \left(\frac{3}{4} \varepsilon A^2 - \omega^2 \right) A \cos s \right) ds \\
 &= A \cos \omega t - \frac{1}{2} \left(\frac{3}{4} \varepsilon A^2 - \omega^2 \right) A t \sin \omega t - \frac{\varepsilon A^3 (\cos \omega t - \cos 3\omega t)}{32\omega^2}.
 \end{aligned}
 \tag{48}$$

Eliminating the secular “ $t \sin t$ ” term needs

$$\omega = \frac{\sqrt{3}}{2} \varepsilon^{1/2} A.
 \tag{49}$$

Its period, therefore, can be written as

$$T = \frac{4\pi}{\sqrt{3}}\varepsilon^{-1/2}A^{-1} = 7.25\varepsilon^{-1/2}A^{-1}. \quad (50)$$

Its exact period can be readily obtained, which reads

$$T_{ex} = 4\sqrt{2} \int_0^{\pi/2} \frac{\sin x dx}{\sqrt{\varepsilon A^2 \sin^2 x (1 + \cos^2 x)}} = \frac{7.4163}{\varepsilon^{1/2}A}. \quad (51)$$

It is obvious that the maximal relative error is less than 2.24%, and the obtained approximate period is valid for all $\varepsilon > 0$.

We can also use the following iteration

$$u_{n+1}(t) = u_0(t) + \int_0^t (s-t)\varepsilon u_n^3 ds. \quad (52)$$

Beginning with $u_0(t) = u(0) + u'(0)t = A$, we obtain its first-order approximate solution which reads

$$u_1(t) = A - \frac{1}{2}\varepsilon A^3 t^2 + O(t^4) \quad (53)$$

$$u_2 = A - \frac{cA^3}{2}t^2 + \frac{c^2A^5}{8}t^4 - \frac{c^3A^7}{40}t^6 + O(t^8). \quad (54)$$

If we set

$$u_1(t_0) = A - \frac{1}{2}\varepsilon A^3 t_0^2 = 0, \quad (55)$$

then its period can be approximately obtained as

$$T = 4t_0 = \frac{4\sqrt{2}}{\varepsilon^{1/2}A} = \frac{5.657}{\varepsilon^{1/2}A} \quad (56)$$

$$u(t) = A \cos \frac{\pi \varepsilon^{1/2} A}{2\sqrt{2}} t. \quad (57)$$

Its accuracy researches 23.7%. The accuracy can be further improved if we set

$$u_2(t_0) = A - \frac{cA^3}{2}t_0^2 + \frac{c^2A^5}{8}t_0^4 - \frac{c^3A^7}{40}t_0^6 = 0, \quad (58)$$

then we have

$$T = 4t_0 = 4\sqrt{\frac{2.8790}{\varepsilon A^2}} = \frac{6.787}{\varepsilon^{1/2}A} \quad (59)$$

$$u(t) = A \cos \left(\frac{\pi \varepsilon^{1/2} A}{2\sqrt{2.8790}} \right) t. \quad (60)$$

Now the 8.5% accuracy is good.

Now we consider a general nonlinear oscillator in form

$$u'' + f(u) = 0, \quad u(0) = A, \quad u'(0) = 0 \quad (61)$$

where $f(u)$ is a nonlinear term, a function of u only, and it requires $f(u)/u > 0$.

The iteration formulation can be constructed as follows:

$$u_{n+1}(t) = u_0(0) + u'(0)t + \int_0^t (s-t)f(u_n)ds. \quad (62)$$

By a simple manipulation, its period can be obtained as follows:

$$T = 4t_0 = 4\sqrt{2A/f(A)}. \tag{63}$$

Its approximate solution reads

$$u(t) = A \cos \frac{2\pi}{T}t. \tag{64}$$

To illustrate the effectiveness and convenience of the formulation, Eq. (63), we give some examples.

Case 1

$$f(u) = \frac{u}{1 + \varepsilon u^2}. \tag{65}$$

Its period can be obtained with ease

$$T = 4t_0 = 4\sqrt{2A/f(A)} = 4\sqrt{2(1 + \varepsilon A^2)}. \tag{66}$$

To check its accuracy, we write down its exact period

$$T = 4\sqrt{\varepsilon} \int_0^A \frac{du}{\sqrt{\ln(1 + \varepsilon A^2) - \ln(1 + \varepsilon u^2)}} = 4\sqrt{\varepsilon} \int_0^A \frac{du}{\sqrt{\ln[(1 + \varepsilon A^2)/(1 + \varepsilon u^2)]}}. \tag{67}$$

In case $\varepsilon A^2 \rightarrow \infty$, the period (67) reduces to

$$\lim_{\varepsilon A^2 \rightarrow \infty} T = 4\sqrt{\varepsilon} \int_0^A \frac{du}{\sqrt{2(\ln A - \ln u)}} = 4\sqrt{2\varepsilon}A \int_0^\infty \exp(-x^2)dx = 2\sqrt{2\pi\varepsilon}A. \tag{68}$$

The accuracy reaches 12.8% even when $\varepsilon A^2 \rightarrow \infty$.

Case 2

$$f(u) = \frac{\varepsilon}{u}. \tag{69}$$

Its period reads

$$T = 4\sqrt{2A/f(A)} = 4\sqrt{2A^2/\varepsilon}. \tag{70}$$

While its exact period is

$$\begin{aligned} T &= 2\sqrt{2} \int_0^A \frac{du}{\sqrt{\int_u^A \varepsilon/s ds}} = \frac{2\sqrt{2}}{\sqrt{\varepsilon}} \int_0^A \frac{du}{\sqrt{\ln A - \ln u}} = \frac{2\sqrt{2}}{\sqrt{\varepsilon}} A \int_0^1 \frac{du}{\sqrt{\ln(1/s)}} \\ &= \frac{2\sqrt{2}}{\sqrt{\varepsilon}} A \times \sqrt{\pi} = 2\sqrt{2\pi/\varepsilon}A. \end{aligned} \tag{71}$$

The accuracy is 12.8%.

Case 3

$$f(u) = \text{sign}(u), \tag{71}$$

where

$$\text{sign}(u) = \begin{cases} 1, & u > 0 \\ -1, & u \leq 0. \end{cases}$$

Its period is

$$T = 4t_0 = 4\sqrt{2A/f(A)} = 4\sqrt{2A} \tag{72}$$

which happens to be the exact period.

Case 4

$$f(u) = \varepsilon u^{2n+1}, \quad T = 4\sqrt{2A/f(A)} = 4\sqrt{\frac{2}{\varepsilon A^{2n}}} \tag{73}$$

$$f(u) = \frac{au^3}{b + cu^2}, \quad T = 4\sqrt{2A/f(A)} = 4\sqrt{\frac{2(b + cA^2)}{aA^2}} \tag{74}$$

$$f(u) = u^{1/3}, \quad T = 4\sqrt{2A/f(A)} = 4\sqrt{2A^{2/3}} \tag{75}$$

$$f(u) = \varepsilon u^{1/(2n+1)}, \quad T = 4\sqrt{2A/f(A)} = 4\sqrt{\frac{2A^{2n/(2n+1)}}{\varepsilon}} \tag{76}$$

$$f(u) = u + u^{1/3}, \quad T = 4\sqrt{2A/f(A)} = 4\sqrt{2/(1 + A^{-2/3})}. \tag{77}$$

Example 4. Consider a nonlinear wave equation

$$u_{tt} - c^2 u_{xx} + f(u, u_t, u_x, u_{xx}, u_{tx}, u_{tt}, u_{xxx}, \dots) = 0 \tag{78}$$

with initial conditions

$$u(x, 0) = F(x) \tag{79}$$

$$u_t(x, 0) = G(x). \tag{80}$$

In case $f = 0$ we have the D’Alembert solution which read

$$u_{n+1}(x, t) = \frac{F(x + ct) + F(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} G(s) ds. \tag{81}$$

So we can construct the following iteration:

$$u_{n+1}(x, t) = \frac{F(x + ct) + F(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} G(s) ds + \int_0^t (s - t)(f_n - c^2(u_n)_{xx}) ds \tag{82}$$

where $f_n = f(u_n, (u_n)_t, (u_n)_x, (u_n)_{xx}, \dots)$.

If we begin with $u_0(x, t) = u(x, 0) + tu_t(x, 0)$, then we have the following approximate solution

$$u(x, t) = \frac{F(x + ct) + F(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} G(s) ds + \int_0^t (s - t) f ds. \tag{83}$$

It is interesting to note that Eq. (83) satisfies Eq. (78) and all initial conditions (79) and (80). Differentiating both sides of Eq. (83) with respect to t and x yields

$$u_t(x, t) = \frac{cF'(x + ct) - cF'(x - ct)}{2} + \frac{G(x + ct) + G(x - ct)}{2} - \int_0^t f ds \tag{84}$$

$$u_{tt}(x, t) = \frac{c^2 F''(x + ct) + c^2 F''(x - ct)}{2} + \frac{cG'(x + ct) - cG'(x - ct)}{2} - f \tag{85}$$

$$u_x(x, t) = \frac{F'(x + ct) - F'(x - ct)}{2} + \frac{G(x + ct) - G(x - ct)}{2c} \tag{86}$$

$$u_{xx}(x, t) = \frac{F''(x + ct) - F''(x - ct)}{2} + \frac{G'(x + ct) - G'(x - ct)}{2c}. \tag{87}$$

From (85) and (87) we have $u_{tt} - c^2 u_{xx} + f = 0$, from (83) and (84) we obtain, respectively, $u(x, 0) = F(x)$, and $u_t(x, 0) = G(x)$. So the iteration formulation (82) can be simplified as

$$u_{n+1}(x, t) = \frac{F(x + ct) + F(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} G(s) ds + \int_0^t (s - t) f_n ds. \tag{88}$$

4. Conclusions

This paper reviews the current status of research and development of variational iteration method; the most useful iteration formulations are listed in a convenient form for later reference and systematic use. Compared to previous convenient tutorial review of the method (He Jh. Variational Iteration Method—Some Recent Results and New Interpretations, Journal of Computational and Applied Mathematics) [9], this paper aims at communicating necessary theoretical background knowledge required for an in-depth study of the variational iteration method and its application.

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