International Journal of Modern Physics B
Vol. 20, No. 18 (2006) 2561–2568

ADDENDUM

NEW INTERPRETATION OF HOMOTOPY PERTURBATION METHOD*

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Received 26 June 2006

The present work constitutes a guided tour through the mathematics needed for a proper understanding of homotopy perturbation method as applied to various nonlinear problems. It gives a new interpretation of the concept of constant expansion in the homotopy perturbation method.

Keywords: Homotopy perturbation method; constant expansion; nonlinear equations; asymptotic solution.

1. Introduction

We consider a general nonlinear oscillator in the form

$$mu'' + \omega_0^2 u + \varepsilon f(u, u', u'') = 0 \tag{1}$$

where m and ω_0^2 are constants, f is a nonlinear term. In case $m\omega_0^2 \leq 0$, the traditional perturbation method does not work. But homotopy perturbation method[1–4] can completely eliminate this limitation by expanding the constants m and ω_0^2 in the following ways:

$$\omega_0^2 = \omega^2 + p\omega_1 + p^2\omega_2 + \cdots, \tag{2}$$

$$m = 1 + pm_1 + p^2 m_2 + \cdots, \tag{3}$$

where p is a homotopy parameter, ω^2, ω_i and m_i are unknown constants to be further determined.

For nth order approximate solution, Eqs. (2) and (3) should be replaced by

$$\omega_0^2 = \omega^2 + p\omega_1 + p^2\omega_2 + \cdots + p^n\omega_n\,, \tag{4}$$

*This paper is a supplement to my review paper "Some asymptotic methods for strongly nonlinear equations" published in IJMPB Vol. 20 No. 10, pp. 1141–1199, 2006.

$$m = 1 + pm_1 + p^2 m_2 + \cdots + p^n m_n \,. \tag{5}$$

Generally we always stop before $n = 2$.

Parameter-expansion appeared in my previous publications.[5–8] Though the technology shows great success in various fields,[9–13] its mathematical rigor is debated recently. A new interpretation of the parameter-expansion is, therefore, much needed.

2. Validity of the Parameter-Expansion

To illustrate its validity of the parameter-expansion, we consider a nonlinear oscillator[4]

$$u'' + \varepsilon u^3 = 0 \,, \qquad u(0) = A \,, \qquad u'(0) = 0 \,. \tag{6}$$

In our study, the parameter ε is not required to be small, $0 < \varepsilon < \infty$.

2.1. *First-order approximate solution*

We construct a homotopy in the form

$$u'' + (\omega^2 + pc_1)u + p\varepsilon u^3 = 0 \,, \qquad u(0) = A \,, \qquad u'(0) = 0 \,, \qquad p \in [0, 1] \tag{7}$$

where

$$\omega^2 + c_1 = 0 \,. \tag{8}$$

It is obvious that when $p = 0$, Eq. (7) becomes a linear equation

$$u'' + \omega^2 u = 0 \,, \qquad u(0) = A \,, \qquad u'(0) = 0 \,, \tag{9}$$

when $p = 1$ Eq. (7) turns out to be the original nonlinear one. The embedding parameter p monotonically increases from zero to unit as the linearized equation (9), which is easy to solve, is continuously deformed to the original nonlinear problem under study. The basic assumption of the homotopy perturbation method is that the solution of Eq. (7) can be written as a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + \cdots \,. \tag{10}$$

Substituting Eq. (10) into Eq. (7), and equating coefficients of like powers of p yields the following equations:

$$u_0'' + \omega^2 u_0 = 0 \,, \qquad u_0(0) = A \,, \qquad u_0'(0) = 0 \,, \tag{11}$$

$$u_1'' + \omega^2 u_1 + c_1 u_0 + \varepsilon u_0^3 = 0 \,, \qquad u_1(0) = 0 \,, \qquad u_1'(0) = 0 \,. \tag{12}$$

Solving Eq. (11), we have

$$u_0 = A \cos \omega t \,. \tag{13}$$

Substituting u_0 into Eq. (12) results in

$$u_1'' + \omega^2 u_1 + A \left(c_1 + \frac{3}{4} \varepsilon A^2 \right) \cos \omega t + \frac{1}{4} \varepsilon A^3 \cos 3\omega t = 0 \,. \tag{14}$$

Eliminating the secular term we need

$$c_1 = -\frac{3}{4}\varepsilon A^2 . \tag{15}$$

If only the first-order approximate solution is searched for, then from Eq. (8), we have

$$\omega = \frac{\sqrt{3}}{2}\varepsilon^{1/2} A . \tag{16}$$

Its period, therefore, can be written as

$$T = \frac{4\pi}{\sqrt{3}}\varepsilon^{-1/2} A^{-1} = 7.25\varepsilon^{-1/2} A^{-1} . \tag{17}$$

Its exact period can be readily obtained, which reads

$$T_{\text{ex}} = 4\sqrt{2}\int_0^{\pi/2} \frac{\sin x dx}{\sqrt{\varepsilon A^2 \sin^2 x(1+\cos^2 x)}} = \frac{6.743}{\varepsilon^{1/2} A} . \tag{18}$$

It is obvious that the maximal relative error is less than 7.5%, and the obtained approximate period is valid for all $\varepsilon > 0$.

Since secular terms arise in u_i $(i \geq 2)$, the constructed homotopy equation (7) is, therefore, only valid for the first-order approximate.

3. Second-order Approximate Solution

If we want to obtain a second-order approximate solution, we have to replace Eq. (7) with one of the following

$$u'' + (\omega^2 + pc_1 + p^2 c_2)u + p\varepsilon u^3 = 0 , \quad u(0) = A , \quad u'(0) = 0 , \quad p \in [0,1] \tag{19}$$

where

$$\omega^2 + c_1 + c_2 = 0 . \tag{20}$$

Proceeding the same way as before, we obtain the following equations:

$$u_0'' + \omega^2 u_0 = 0 , \qquad u_0(0) = A , \qquad u_0'(0) = 0 , \tag{21}$$

$$u_1'' + \omega^2 u_1 + c_1 u_0 + \varepsilon u_0^3 = 0 , \tag{22}$$

$$u_2'' + \omega^2 u_2 + c_1 u_1 + c_2 u_0 + 3\varepsilon u_0^2 u_1 = 0 . \tag{23}$$

The initial conditions for u_1 and u_2 should satisfy $u_1(0) + u_2(0) = 0$ and $u_1'(0) + u_2'(0) = 0$.

The solution for u_0 is $u_0 = A \cos \omega t$, and a particular solution of Eq. (22) reads

$$u_1(t) = \frac{\varepsilon A^3}{32\omega^2} \cos 3\omega t . \tag{24}$$

Substituting u_0 and u_1 into Eq. (23), and simplifying the resulted equation, we have

$$u_2'' + \omega^2 u_2 + A\left(c_2 + \frac{3\varepsilon^2 A^4}{128\omega^2}\right)\cos\omega t + \left(\frac{\varepsilon A^3 c_1}{32\omega^2} + \frac{3\varepsilon^2 A^5}{64\omega^2}\right)\cos 3\omega t + \frac{3\varepsilon^2 A^5}{128\omega^2}\cos 5\omega t = 0\,. \tag{25}$$

No secular term in u_2 requires

$$c_2 = -\frac{3\varepsilon^2 A^4}{128\omega^2}\,. \tag{26}$$

In view of Eq. (20), we have

$$\omega^2 - \frac{3}{4}\varepsilon A^2 - \frac{3\varepsilon^2 A^4}{128\omega^2} = 0\,, \tag{27}$$

which leads to the result

$$\omega = 0.8832022\varepsilon^{1/2}A\,. \tag{28}$$

Now the accuracy of frequency reaches 5.2%.

3.1. *nth order approximate solution*

Secular terms arise in u_i $(i \geq 3)$ in homotopy equation (19). To search for nth order approximate solution, we re-write Eq. (19) in the form

$$u'' + (\omega^2 + pc_1 + p^2c_2 + p^3c_3 + \cdots + p^n c_n)u + p\varepsilon u^3 = 0\,, \quad u(0) = A\,, \quad u'(0) = 0\,. \tag{29}$$

where the unknown constants c_i can be identified in view of no secular terms in u_i $(i = 1, 2, 3, \ldots, n)$.

The solution procedure is equivalent to my previous treatment. We can re-write Eq. (6) in the form

$$u'' + 0 \cdot u + p\varepsilon u^3 = 0\,, \qquad u(0) = A\,, \qquad u'(0) = 0\,. \tag{30}$$

If nth order approximate solution is searched for, we assume that the solution and the constant, zero, in Eq. (30) can be expressed in the forms

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots + p^n u_n\,, \tag{31}$$

$$0 = \omega^2 + pc_1 + p^2c_2 + p^3c_3 + \cdots + p^n c_n\,. \tag{32}$$

We can obtain the same differential equations for u_i $(i = 0, 1, 2, 3, \ldots, n)$ as those illustrated above.

4. Discussions

4.1. *Asymptotic character of the homotopy perturbation method*

Homotopy perturbation method is a kind of asymptotic methods, though the higher-order approximate solution leads to higher accuracy of the period, and the error for amplitude might become larger. In case the amplitude does not vary with time as illustrated in the above example, we always use zero-order approximate solution:

$$u(t) = A \cos\left(\frac{\sqrt{3}}{2}\varepsilon^{1/2}A\right)t\,, \tag{33}$$

or

$$u(t) = A \cos(0.8832\varepsilon^{1/2}A)t\,. \tag{34}$$

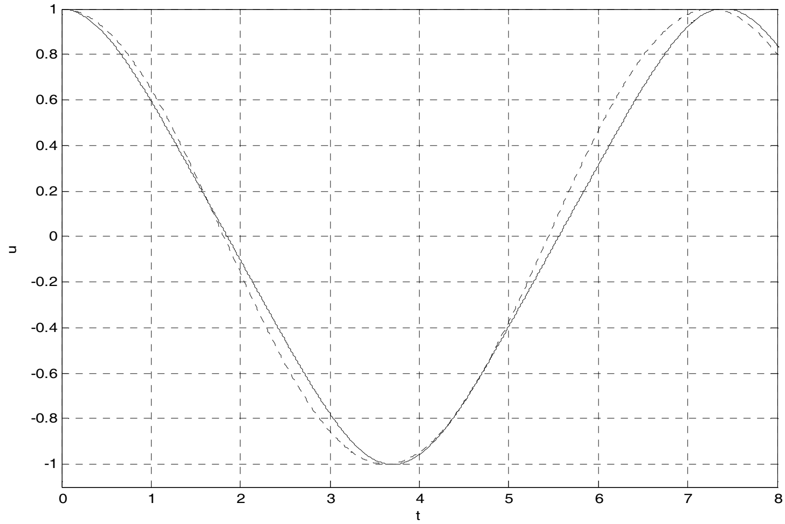
Comparison of approximate solution, Eq. (33), with the exact solution is shown in Fig. 1.

4.2. *Mathematical exactness of the parameter expansion*

In homotopy perturbation method, a constant can be expanded into a series of the homotopy parameter, for example,

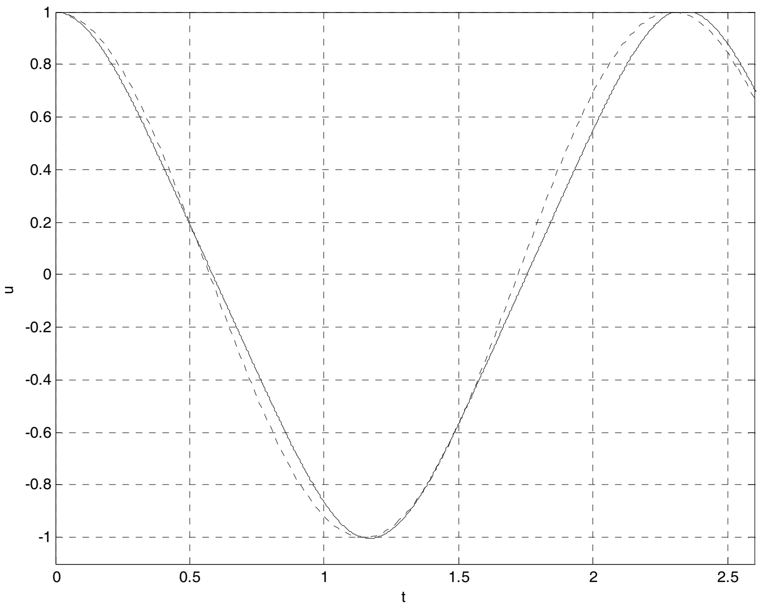
$$0 = \omega^2 + pc_1 + p^2c_2 + p^3c_3 + \cdots\,, \qquad p \in [0,1]\,. \tag{35}$$

If Eq. (35) is considered as an equality holding for all p, according to Fitzpatrick's theorem,[14] we have $\omega^2 = 0$ and $c_i = 0$ for all i.

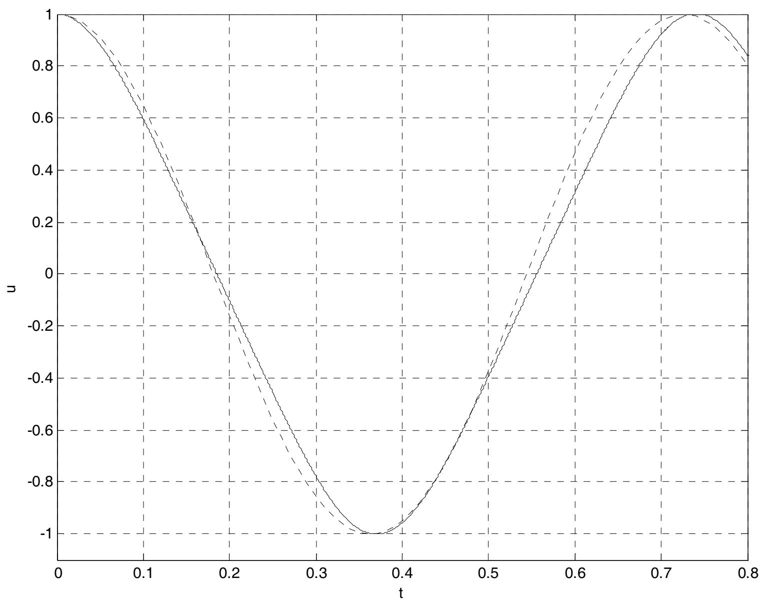


(a) $\varepsilon = 1, A = 1$

Fig. 1. Comparison of approximate solution, Eq. (33), with exact solution. Dashed line: exact solution, continued line: approximate solution.

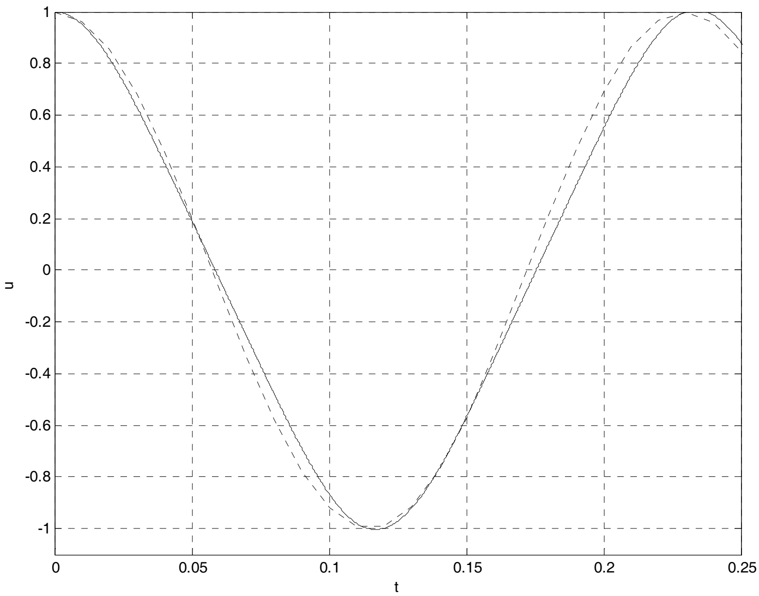


(b) $\varepsilon = 10, A = 1$



(c) $\varepsilon = 100, A = 1$

Fig. 1. (*Continued*)



(d) $\varepsilon = 1000, A = 1$

Fig. 1. (*Continued*)

Actually we did not search for an infinite order approximate solution, we always stop before $i = 2$, so we need not guarantee the convergence of the series. Actually Eq. (35) can be an asymptotic series. We can consider Eq. (35) as an expansion which holds for all variables that lie in a certain set $p \in [0, 1]$.

5. Conclusions

Homotopy perturbation method is a relatively new method, it is still evolving. Like other methods, it has theoretical and application limitations.

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