

Econ 240A - Part II - Fall 2005

Section 2: Linear Algebra Review

Maria Carolina Caetano

Friday, November 4

1 Vector Space:

Definition 1 A vector space is a set E , whose elements are called vectors, where two operations are defined:

- Sum: $v + u \in E$
- Scalar Multiplication: $\alpha v \in E, \forall \alpha \in \mathfrak{R}$

These operations have to satisfy certain properties to be called "sum" and "scalar multiplication":

- $u + v = v + u$
- $(u + v) + w = u + (v + w)$, and $(\alpha\beta)v = \alpha(\beta v)$
- \exists a vector $0 \in E$, called zero vector, such that $v + 0 = 0 + v = v$, for all $v \in E$.
- $\forall v \in E$, there exists a vector $-v \in E$, such that $-v + v = v + (-v) = 0$
- $(\alpha + \beta)v = \alpha v + \beta v$, and $\alpha(u + v) = \alpha u + \alpha v$
- $1 \cdot v = v$

Example 2

- \mathfrak{R} , $u = a, v = b, u + v = a + b, \alpha v = \alpha b$
- $\mathfrak{R}^n, u = (a_1, \dots, a_n), v = (b_1, \dots, b_n), u + v = (a_1 + b_1, \dots, a_n + b_n), \alpha v = (\alpha b_1, \dots, \alpha b_n)$
- \mathfrak{F} space of functions, $(f + g)(x) = f(x) + g(x)$;
- \mathcal{L} space of random variables defined in Ω with values in \mathfrak{R} .
 $(X + Y)(\omega) = X(\omega) + Y(\omega), (\alpha X)(\omega) = \alpha X(\omega)$.

2 Linear Dependence

Definition 3 A set of vectors $\{v_1, \dots, v_n\}$ is said to be linearly independent if:
 $\alpha_1 v_1 + \dots + \alpha_n v_n = 0 \implies \alpha_1 = \dots = \alpha_n = 0$.

Theorem 4 $v = \alpha_1 v_1 + \dots + \alpha_n v_n$, and $v = \beta_1 v_1 + \dots + \beta_n v_n \implies \alpha_1 = \beta_1, \dots, \alpha_n = \beta_n$. This means that the representation of a vector as a linear combination of linearly independent vectors is unique.

Theorem 5 Let $\{v_1, \dots, v_n\}$ be a set of vectors. If none of them can be expressed as a linear combination of the previous, then $\{v_1, \dots, v_n\}$ is LI. So, if

$$\begin{aligned} v_1 &\neq 0 \\ v_2 &\neq \alpha v_1, \forall \alpha \in \mathfrak{R} \\ v_3 &\neq \alpha v_1 + \beta v_2, \forall \alpha, \beta \in \mathfrak{R}, \\ &\dots \end{aligned}$$

Then, $\{v_1, \dots, v_n\}$ is LI.

Definition 6 A set of vectors $B = \{v_1, \dots, v_n\}$ is said to be a basis of the vector space E if:

- $\{v_1, \dots, v_n\}$ is LI.
- $\forall v \in E$, there exist scalars $\alpha_1, \dots, \alpha_n$ such that $v = \alpha_1 v_1 + \dots + \alpha_n v_n$. This property is expressed as $\{v_1, \dots, v_n\}$ "generates" E , or $\{v_1, \dots, v_n\}$ "spans" E .

Example 7

- $\mathfrak{R}, B = \{1\}$
- $\mathfrak{R}^n, B = \{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}$
- \wp^n space of polynomials of degree n . $B = \{1, x, x^2, \dots, x^n\}$
- \wp space of all polynomials. $B = \{1, x, x^2, x^3, \dots\}$
- $M(k \times n)$ space of $k \times n$ matrices. $B = \{A_1, \dots, A_{kn}\}$, where

$$A_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}, \dots, A_k = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Definition 8 We say that the dimension of a space is the number of elements of its base.

- $\dim(\mathfrak{R}) = 1$
- $\dim(\mathfrak{R}^n) = n$
- $\dim(\wp^n) = n + 1$

- $\dim(\varphi) = \infty$
- $\dim(M(k \times n)) = kn$

Theorem 9 If $\{v_1, \dots, v_n\}$ generates E , then any set with more than n vectors is LD (linearly dependent).

3 Linear Transformations

Definition 10 A linear transformation $A : E \rightarrow F$, is a function, such that:

$$A(u + v) = Au + Av$$

$$A(\alpha v) = \alpha v$$

Example 11

- $A : \mathfrak{R}^2 \rightarrow \mathfrak{R}^3$, such that let $v = (x, y)$,
- $Av = (x + y, x - y, 3x)$
- We can represent A as a matrix:
- $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 3 & 0 \end{bmatrix}$, so $Av = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 3 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - y \\ 3x \end{pmatrix}$
- \mathcal{L} space of integrable functions: $A : \mathcal{L} \rightarrow R$; $Af = \int f(x)dx$. Check that this is a linear transformation.
- L^1 space of random variables with finite expectation. $A : L^1 \rightarrow R$; $AX = E(X)$. Check that this is also a linear transformation.

4 Inner Product

Definition 12 Inner product is a function $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathfrak{R}$ which satisfies the properties:

- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
- $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$
- $\langle u, v \rangle = \langle v, u \rangle$
- $\langle u, u \rangle \geq 0, \langle u, u \rangle = 0 \Leftrightarrow u = 0$

Example 13

- $\mathfrak{R}, u = a, v = b, \langle u, v \rangle = ab$
- $\mathfrak{R}^n, u = (a_1, \dots, a_n), v = (b_1, \dots, b_n), \langle u, v \rangle = a_1b_1 + \dots + a_nb_n$
- \mathcal{L} space of square integrable functions, $\langle f, g \rangle = \int f(x)g(x)dx$
- L^2 space of random variables with $E(X^2) < \infty, \langle X, Y \rangle = E(XY)$

Example 14 *Definition 15* A norm is a measure of size in a vector space with an inner product. It is a function $\|\cdot\| : E \rightarrow \mathfrak{R}^+$, which satisfies the properties:

a) $\|v\| \geq 0, \|v\| = 0 \Leftrightarrow v = 0$

b) $\|\alpha v\| = |\alpha| \|v\|$

c) $\|u + v\| \leq \|u\| + \|v\|$

In general, we can define a norm by doing $\|v\| = \sqrt{\langle v, v \rangle}$

Example 16

- $\mathfrak{R}, v = a, \|v\| = |a|$

- $\mathfrak{R}^n, v = (a_1, \dots, a_n), \|v\| = \sqrt{a_1^2 + \dots + a_n^2}$

Observe the other norms can be defined:

$$\|v\| = |a_1| + \dots + |a_n|$$

$$\|v\| = \max\{|a_1|, \dots, |a_n|\}$$

Check that they satisfy the properties of a norm.

- \mathcal{L} space of square integrable functions, $\|f\| = \int f(x)^2 dx$

- L^2 space of random variables with $E(X^2) < \infty$. $\|X\| = \sqrt{E(X^2)}$

Definition 17 Two vectors u, v are said to be orthogonal if $\langle u, v \rangle = 0$, and are denoted $u \perp v$

Theorem 18 Every set of orthogonal vectors is LI.

5 Projection

Definition 19 A projection is a linear transformation $P : E \rightarrow E$ (operator) such that $P^2 = P$.

This means that for any vector v , $P^2v = P(Pv) = Pv$.

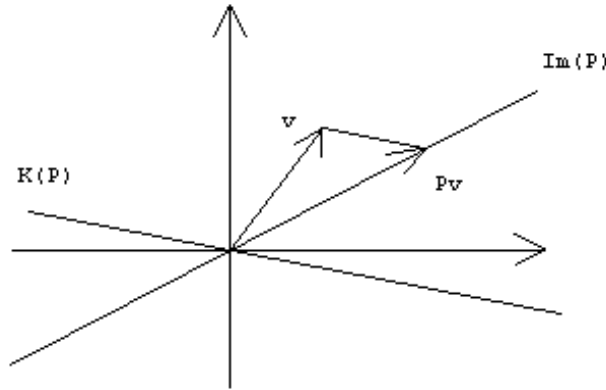
The image $\text{Im}(P)$ of a projection is the subspace of E formed by all vectors $u = Pv$

The kernel $K(P)$ of a projection is the subspace of E of all vectors v such that $Pv = 0$

We say that "P projects any vector from E onto $\text{Im}(P)$ parallel to $K(P)$."

Definition 20 An orthogonal projection is a projection where $\text{Im}(P) \perp K(P)$. This means that every vector which is orthogonal to $\text{Im}(P)$ is annihilated.

Example 21 In \mathfrak{R}^n , the orthogonal projection of a vector u onto the subspace generated by a vector v has an interesting formula: $P_v(u) = \frac{\langle u, v \rangle}{\langle v, v \rangle} v$



6 Eigenvalues and Eigenvectors

Definition 22 Let $A : E \rightarrow E$ be an operator. If there exists a vector v and a constant λ such that $Av = \lambda v$, then v is called an eigenvector of A , with correspondent eigenvalue λ .

Theorem 23 If λ_i, λ_j are eigenvalues of the same operator A , $\lambda_i \neq \lambda_j$, then the corresponding eigenvectors v_i and v_j are LI.

7 Symmetric Operators

Definition 24 An operator $A : E \rightarrow E$ is said to be symmetric if

$$\langle Au, v \rangle = \langle u, Av \rangle, \forall u, v \in E.$$

In \mathbb{R}^n , all operators are represented by matrices $n \times n$, and A is symmetric if it's $n \times n$ corresponding matrix is symmetric. Let A' be the transpose of the matrix A , the property of symmetry is the same as $A' = A$.

Theorem 25 Any orthogonal projection is a symmetric operator.

Theorem 26 If λ_i, λ_j are eigenvalues of the same symmetric operator A , $\lambda_i \neq \lambda_j$, then the corresponding eigenvectors v_i and v_j are orthogonal: $v_i \perp v_j$.

Theorem 27 (Spectral Theorem) Let $A : E \rightarrow E$ be a symmetric operator. Then there exists a basis B of E formed by orthogonal eigenvectors of A .

If we are in \mathbb{R}^n , there exist an $n \times n$ matrix with orthogonal columns of norm 1 (the columns are the orthogonal eigenvectors of the basis, divide each element by its norm, so the resulting new base has all orthogonal eigenvectors of size 1.) This matrix is denoted Q and has the property that $Q' = Q^{-1}$.

Then, because of this theorem, we can write $QAQ' = D$, where D is a diagonal matrix, with the eigenvectors correspondent to each column in the diagonal.

Definition 28 A symmetric operator is called positive if $\langle Au, u \rangle \geq 0, \forall u \in E$

Observe that for any transformation $X : E \rightarrow F$, the transformation $X'X : E \rightarrow E$ is clearly symmetric, since $(X'X)' = X'X$. Moreover, $X'X$ is also a positive operator¹.

Theorem 29 If A is a positive operator, all its eigenvalues $\lambda_1, \dots, \lambda_p \geq 0$

If A is a symmetric operator, we know we can write $QAQ' = D$, because of the Spectral Theorem. However, the diagonal matrix has a non-negative diagonal. We can therefore do very interesting manipulations with this matrix, and therefore express A as a $X'X$ kind of matrix.

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix}$$

$$= \sqrt{D}\sqrt{D}$$

$$\text{Hence, } A = Q'DQ = Q'\sqrt{D}\sqrt{D}Q = Q'\sqrt{D}'\sqrt{D}Q = (\sqrt{D}Q)'(\sqrt{D}Q).$$

¹ $\langle X'Xu, u \rangle = \langle Xu, Xu \rangle \geq 0$