Comparative Statics of General Equilibrium Asset Prices

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Abstract

I study comparative statics of asset prices in a representative-agent model where dividends are vector-geometric Brownian motions. Due to wealth effects, the equilibrium relative price of a security may vary with the current realization of a component of the Brownian vector even when its dividend is independent of that component. I examine analytically an element of wealth effects that has hitherto been ignored by the literature. Changes in wealth do not operate only through changes in risk aversion. They alter also the riskiness of a security by changing the correlation between its payoff and the marginal utility of equilibrium consumption. This enhances the extent to which market-clearing leads to endogenously-generated correlation across asset prices and returns, over and above that induced by correlation between payoffs, giving the appearance of "contagion".

Keywords: asset pricing, comparative statics, contagion, dynamically-complete markets. **JEL Classification Numbers**: G10, G12.

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1 Introduction

Consider a representative-agent model where uncertainty is described by a standard d-dimensional Brownian motion $\beta = (\beta_1, ..., \beta_d)' \in \mathbb{R}^d$ with respect to a filtration $\{\mathcal{F}_t : t \in [0, T]\}$ on a probability space $(\Omega, \mathcal{F})$. There are $J + 1$ assets: a zero-coupon bond B and J risky securities, indexed by $j \in \{1, \ldots, J\}$ and specified by their respective dividend processes $A_j(\omega, t)$. The dividend specification is as in Raimondo [32] and Anderson and Raimondo [4]:

$$A_j(\omega, t) \equiv 0 \quad \text{for } t < T, \qquad A_j(\omega, T) = e^{\mu_j T + \sigma_j \beta(\omega, T)}$$

where σ_j is the row-vector of the (constant) factor loadings of the jth security and $\mu_j \in \mathbb{R}$. As is well-known, in this setting, the equilibrium price $p_{A_j}(\omega, t)$ of the jth security is the conditional expectation of its future dividend valued at the equilibrium marginal utility of the agent.[1] Of course, marginal utilities are not observable in practice and securities are priced with respect to a numeraire, such as dollars. Here, consumption C will be taken as the numeraire: $p_C(\omega, t) = 1 \; \forall t \in [0, T)$.[2]

For the equilibrium price of the jth security relative to that of the bond $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$, what are its dynamics with respect to changes in the (currently observed) component $\beta(\omega, t)$ of the underlying Brownian process? Whether these dynamics are monotone is the most fundamental comparative statics question. For if so, the equilibrium relative prices of the securities vary predictably in response to changes in current information about future dividends. This would, for instance, greatly facilitate econometric analysis since the realized path of the underlying stochastic process (representing the primitive sources of uncertainty) could be

[1] Derivations are provided, for example, in Cox *et al.* [17], for the dynamically-complete, and in Raimondo [32], for the dynamically-incomplete case. With multiple agents, the pricing formula takes the same basic form: the marginal utilities are taken at the equilibrium consumptions of the agents which are determined endogenously as part of the equilibrium (Duffie and Zame [20]; Anderson and Raimondo [5]).

[2] The choice of numeraire is essentially arbitrary because the equilibrium market-clearing condition depends only on the relative prices of the securities and consumption (and does so node (ω, t) by node, not across nodes).

recovered from the path of equilibrium relative asset prices.

Even though the equilibrium relative price process $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ can be derived in closed form in this setting, determining its basic comparative statics' properties is not straightforward for two reasons. First, by the quotient rule, the derivative $\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)$ $k \in \{1,...,d\}$ will be given as the sum of two terms which may well be of opposite sign. Second, and more important, even $p_{A_j}(\omega,t)$ itself may exhibit complex dynamics. An increase in the terminal-period dividend $A_j(\omega,T)$ increases consumption, reducing marginal utility. Since $p_{A_j}(\omega,t)$ is given by the expectation of the product of $A_j(\omega,T)$ with marginal utility, it need not increase when the dividend increases.[3]

My analysis begins by deriving the inner-product of the vector of factor loadings σ_j' with the gradient vector $\nabla_{\beta(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)$. This inner-product is non-negative; more precisely, it is positive unless $\sigma_j' = \mathbf{0}$ (Theorem 2.1, Corollary 2.1). It follows immediately that, if the dividend $A_j(\omega,T)$ depends on only one component, say $\beta_m(\omega,T)$, of the Brownian vector, $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ is monotone in $\beta_m(\omega,t)$. Specifically, $\frac{\partial}{\partial\beta_m(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)$ has the same sign as the factor loading σ_{jm} (Corollary 3.1). This applies on the entire set of securities in two important cases: when there is a single source of uncertainty affecting dividends ($d = 1$) or when the matrix of factor loadings σ is diagonal.[4] For a more general setting, Proposition 4.2.2 establishes monotonicity when the agent exhibits constant relative risk aversion (CRRA), has no terminal-period endowment ($\rho(\omega,T) = 0$, $\forall\omega \in \Omega$), and σ is such that $\sigma_i\sigma_j' = \sigma_j\sigma_j'$ for every security i. In this case, $\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)$ has always the same sign as σ_{jk}.

I proceed to study the derivative $\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)$ when the terminal-period dividend $A_j(\omega,T)$ is independent of the $\beta_k(\omega,T)$ component of the Brownian vector ($\sigma_{jk} = 0$). To the extent that $\beta_k(\omega,T)$ does affect some of the other $J-1$ terminal-period dividends or the

[3]See Raimondo [32] for an example, with log-utility, in which $p_{A_j}(\omega,t)$ is *independent* of $A_j(\omega,T)$. All of the adjustment in the relative price, that is needed to clear the markets, occurs in $p_B(\omega,t)$.

[4]The $J \times d$ matrix σ has the row-vector σ_j as its jth row. A diagonal matrix σ is necessarily square: $J = d$, $\sigma_{kk} \neq 0$, and $\sigma_{jk} = 0$ for $j \neq k$ with $j,k \in \{1,...,d\}$.

agent's terminal-period endowment, it induces wealth effects which may require adjustments in the equilibrium relative price of the jth security. Propositions 4.1.4 and 4.2.1 give the general expressions for $\frac{\partial}{\partial \beta_k(\omega,t)} \left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)} \right)$ when $\sigma_{jk} = 0$. Evidently, the dynamics of the equilibrium relative price are rich and the cross-derivative will not be zero apart from quite unusual cases. The comparative statics of equilibrium relative prices are complex since changes in the underlying Brownian process induce wealth effects which alter not only the agent's risk aversion but also the "riskiness" of a security. The latter effect operates through changing the covariance between the dividend of the security and the marginal utility of the agent. It will be referred to, henceforth, as the asset-riskiness effect and described in detail in Section 2.

The complexity of the dynamics of relative asset prices can be demonstrated analytically in some settings. Propositions 4.1.1 and 4.1.2 indicate how unusual are the situations in which $\frac{\partial}{\partial \beta_k(\omega,t)} \left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)} \right) = 0$ when $\sigma_{jk} = 0$. The agent exhibiting constant absolute risk aversion (CARA) along with a diagonal matrix of factor loadings σ do not suffice. We also need $\beta_k(\omega, T)$ and the component of the Brownian vector that determines $A_j(\omega, T)$ to affect the agent's terminal-period wealth independently. Proposition 4.1.3 provides an example of the derivative in question being non-zero when $\sigma_{jk} = 0$ under CARA. Specifically, the dividend $A_j(\omega, T)$ depends only on the mth ($m \neq k$) component of the Brownian vector, $\beta_m(\omega, T)$, there exists another security ($l \neq j$) whose terminal-period dividend $A_l(\omega, T)$ varies only with $\beta_k(\omega, T)$ and $\beta_m(\omega, T)$, and $\beta_k(\omega, T)$ affects no component of terminal-period wealth other than $A_l(\omega, T)$. In this case, $\frac{\partial}{\partial \beta_k(\omega,t)} \left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)} \right)$ has the opposite sign of σ_{lk}. As long as σ_{lk} is non-zero, therefore, so will be the cross-derivative even though the dividend $A_j(\omega, T)$ is independent of $\beta_k(\omega, t)$ and the risk-aversion channel of wealth effects leaves relative prices unchanged. Regarding more general specifications of risk aversion, Proposition 4.1.5 considers the agent exhibiting decreasing absolute risk aversion (DARA), $J = d = 2$ with a diagonal matrix σ, and constant terminal-period endowment ($\nabla_{\beta(\omega,T)} \rho(\omega, T) = \mathbf{0}$, $\forall \omega \in \Omega$).

In this case, $\frac{\partial}{\partial \beta_k(\omega,t)} \left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)} \right)$ has the same sign as σ_{kk}.

The possibility for a "common factor" or "contagion" in relative asset prices (and, thus, returns) to emerge, when there is no common factor in cash flows, is well-known but has not been demonstrated before analytically in a general equilibrium model. It is noted, for example, in Raimondo [32] and Anderson and Raimondo [4] but no formula is given for the cross-derivative. Kodres and Pritsker [25], Kyle and Xiong [26], and Lagunoff and Schreft [27] show that "contagion" can obtain as a wealth effect in rational expectations equilibria. These are not general equilibrium models, however, as some market participants are not rational (the former two models require the presence of noise traders; the latter of irrational ones). Similarly, Aliprantis *et al.* [1] establish contagion equilibria in a monetary model where players act strategically.

The paper that is closest to the present is Cochrane *et al.* [16] who study asset-price and return dynamics in a representative-agent model with two Lucas [29] trees. Each tree's dividend stream follows a geometric Brownian motion while the agent has log-utility and consumes the sum of the two dividends. This study encompasses a large collection of variables of interest with closed-form solutions given for absolute prices, expected returns, volatilities, correlations, etc. However, the solution method cannot be applied beyond log-utility and two trees (it depends fundamentally upon the dividend-consumption share being the unique state variable) while the dynamics are examined numerically. More importantly, they are given with respect to the dividend-share rather than the underlying stochastic process representing the uncertainty.

Through numerical estimation, the authors show that the prices and returns of the two assets can be positively contemporaneously correlated even though the underlying dividends are independent. Proposition 4.1.5 of this paper confirms the positive correlation of relative prices, analytically and for any DARA utility. The comparative statics with respect to the dividend-share seem to express the underlying market-clearing intuition. If there is a dividend shock, the representative agent wants to rebalance, to spread some of her now larger wealth across both trees. Since she has to hold the fixed assets supply in equilibrium, however, she cannot rebalance, so asset prices and expected returns must adjust. Given a positive dividend shock on tree one, the dividend-share of asset one increases while that of asset two falls. For the agent to become willing to hold asset two in smaller proportion, it must be made less attractive. That is, its price must rise so that its expected return falls. Equivalently, trying but not being able to rebalance some of her larger wealth away from asset one, the agent pushes up the price of asset two.

Although correct, this intuition fails to distinguish between two separate channels through which shocks to current wealth affect market clearing, via changing the agent's risk aversion but also through altering her perception of the "riskiness" of a security. The dynamics of the former mechanism are well-known and straightforward. Those of the latter have not, to the best of my knowledge, hitherto been analyzed by the finance literature and are complex. Under DARA and independent dividend streams, the two mechanisms operate in the same direction which, however, is not universally the case. The operation of the asset-riskiness effect on relative prices can be isolated under CARA since the risk-aversion channel of wealth effects leaves relative prices unchanged. As Proposition 4.1.3 shows, it can easily lead to negative correlation.

The comparative statics formulae and results in this paper lend themselves easily to empirical testing; they can be calculated numerically for any sets of the parameters of the model. The fact that the equilibrium relative prices of assets and asset returns should be correlated, even when their underlying dividends are independent, has important implications for asset-pricing. In particular, it raises questions about the large body of work that focusses on partial equilibrium analysis, treating a small number of securities in isolation from the rest of the market or modeling the equilibrium price process of an asset as a relation that depends only on those sources of uncertainty that directly affect its payoff. Even though the empirical finance literature has focused attention on contagion across national or regional stock markets, a few papers have established that the magnitude of the correlations across asset returns cannot be explained by covariances between the sources of uncertainty that determine their respective payoffs alone. For example, Gropp and Moerman [23] identify within country contagion among large European bank stocks. Driessen *et al.* [18], Lopez and Walter [28], and Moskowitz [31] find evidence that the risk premia are better represented by covariances with the implied market- than by own-variances.

In the model examined here, correlations across asset returns arise endogenously and are stochastic, even though the covariance coefficients of the dividends are constant. There is ample evidence in the empirical finance literature that correlations across asset returns are stochastic. Bollerslev *et al.* [11] present reasonable estimates for a trivariate (U.S. Treasury bills, bonds, and stocks) CAPM model in support of the conclusion that the conditional covariance matrix of asset returns is strongly autoregressive. Other studies have documented cross-sectional relations between risk (measuring generally a stock's risk as the covariance between its return and one or more variables) and expected returns on common stocks. For example, the expected return on a stock has been found to be related to covariances between its return and (i) the return on the market portfolio (Black *et al.* [10], Fama and Macbeth [21]), (ii) factors extracted from multivariate time series of returns (Roll and Ross [34]), (iii) macroeconomic variables (Chen *et al.* [15]), and (iv) aggregate consumption (Breeden *et al.* [13]). See also Andersen *et al.* [3], Alizadeh *et al.* [2], Bansal and Yaron [6], Bansal *et al.* [7], Tauchen [37], Brandt and Diebold [12], and Schwert and Seguin [36] for more recent work.

My analysis contributes also to the literature on equilibrium in continuous-time finance models with a single agent. Existence has been established in a number of papers (Bick [9], He and Leland [24], Cox *et al.* [17], Duffie and Skiadas [19], Raimondo [32]-[33]) but very little is known regarding the question of whether the equilibrium is dynamically complete, if it exists. Apart from Raimondo [32]-[33], all papers deal only with the case in which markets are potentially dynamically-complete ($J = d$).[5] When there are more than one sources of uncertainty ($d > 1$), they show existence of equilibrium by computing the candidate equilibrium price process explicitly and checking that it is dynamically-complete. However, such computation can be done usually only for a very small set of parameter values (in fact, none of the results rules out the possibility that this set has Lebesgue measure zero).[6]

For $J = d$, I show that the equilibrium pricing process is in fact dynamically-complete, for any parameter values, in some important cases. When there is a single source of uncertainty ($d = 1$), $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ is always monotone in $\beta(\omega, t)$. Even though widely asserted, this has not been shown before in the literature. For $d > 1$, Example 4.1.1 demonstrates the equilibrium price process to be dynamically-complete when the matrix of factor loadings σ is diagonal, the terminal-period endowment is constant, and the agent exhibits CARA. In Example 4.2.1, on the other hand, the primitives are taken as in Proposition 4.2.2: the agent exhibits CRRA and has no terminal-period endowment while the matrix σ is such that $\sigma_i \sigma_j' = \sigma_j \sigma_j'$ for every security i. Dynamic completeness obtains as long as σ is non-singular.[7]

To complete introducing my results, it should be noted that there is nothing pathological about the model I study. The utility functions can include any twice continuously-differentiable, state-independent functions representing non-satiated preferences and risk-aversion. The dividend processes are geometric Brownian motions. Both are central benchmarks in continuous-time finance.[8] The agent is endowed with a flow rate of consumption

[5] Raimondo [32]-[33] derives the equilibrium price process directly from the primitives of the economy and in closed form. His existence theorem and equilibrium pricing formulae apply equally well when markets are potentially dynamically-complete ($J = d$) and when they are necessarily dynamically-incomplete ($J < d$).

[6] The issue of dynamic completeness of the equilibrium price process is even more serious in the continuous-time models with many agents. Apart from Anderson and Raimondo [5], the literature assumes, in various forms, that the candidate equilibrium price process is dynamically-complete and proceeds to show that is in fact an equilibrium. See Anderson and Raimondo [5] for a detailed discussion.

[7] Anderson and Raimondo [5] derive existence of equilibrium from the primitives of a continuous-time model with potentially dynamically-complete markets and many agents. They prove also that the candidate equilibrium price process is dynamically-complete, as long as σ is nonsingular, under mild assumptions on the primitives of the model. Their result is obviously more general but their proofs use non-standard analysis. Within its constraints on the primitives, Example 4.2.2 confirms their result deploying standard mathematical apparatus.

[8] Some quite general state-dependence of the utility functions can also be allowed, as long as the depen-

on $[0,T)$ and a lump sum at T (described in many models as a bequest). All J risky securities pay a lump dividend, in units of consumption, at time T (which can be viewed also as the present value of their stream of future dividends in an infinite-horizon framework). These elements define a family of continuous-time securities markets that has been one of the standards in the literature (see, for example, Merton [30]). The particular economy examined here allows for completeness of exposition without imposing significant limitations on the analysis. Within the family of markets described above, the equilibrium relative price process will have to be of the same qualitative form as the one whose comparative statics I examine.

The remaining of the paper is organized as follows. Section 2 overviews the results while Sections 3 and 4 discuss them in detail. Section 5 presents the mathematical derivation of Theorem 2.1. The remaining proofs are to be found in Section 6.

2 Comparative Statics

Consider a representative agent economy in which trade and consumption occur over the compact time interval $[0,T]$.[9] This interval is endowed with a measure λ such that it agrees with the Lebesgue measure on $[0,T)$ while $\lambda(\{T\}) = 1$. The information structure is represented by a filtration $\{\mathcal{F}_t : t \in [0,T]\}$ on a probability space $(\Omega, \mathcal{F}, \mu)$ and a standard d-dimensional Brownian motion $\beta = (\beta_1, \ldots, \beta_d)^T \in \mathbb{R}^d$. The agent has an additively-separable, time-independent utility function. Given a measurable consumption function $c : \Omega \times [0,T] \to \mathbb{R}_{++}$, her utility function is given by

$$U(c) = \mathbb{E}_\mu \left[\int_0^T \varphi_1(c_t)\,\mathrm{d}t + \varphi_2(c_T) \right]$$

dence enters through the process $\{(t, \beta(\omega, t))\}$; that is, as long as it is measurable in the components of the Brownian vector.

[9]The ensuing description borrows heavily from Raimondo [32] as well as Anderson and Raimondo [4].

where the twice continuously-differentiable functions $\varphi_i : \mathbb{R}_{++} \to \mathbb{R}$, $i \in \{1,2\}$, satisfy $\varphi_i'(\cdot) > 0$ and $\varphi_i''(\cdot) < 0$. Her endowment process is constant except for the terminal period T. It is given by $e : \Omega \times [0,T] \to \mathbb{R}_+$, with $e(\omega,t) = 1$ for any $(\omega,t) \in \Omega \times [0,T)$ and $e(\omega,T) = \rho(\beta(\omega,T))$ for some continuous function $\rho : \mathbb{R}^d \to \mathbb{R}_+$.

There is a "bond" (in zero net supply) which pays in units of consumption. Its payoff is given by $B : \Omega \times [0,T] \to \mathbb{R}_+$ with $B(\omega,T) = 1$ and $B(\omega,t) = 0$ at any $t \in [0,T)$. There are also J securities (each in net supply of one unit) with $J \leq d$. Each stock pays off only at time T and the terminal-period dividends follow simple geometric Brownian motions. Taking σ to be a $J \times d$ matrix with σ_j its jth row, let the dividend of the jth stock be $A_j(\omega,T) = e^{\mu_j T + \sigma_j \beta(\omega,T)}$: $\mu_j \in \mathbb{R}_+$, $\sigma_j' \in \mathbb{R}^d$. Raimondo [32] shows that there exists an equilibrium price process for this economy.[10] It can be obtained in terms of the agent's utility function, her terminal-period endowment, and the current realization of the sources of uncertainty as given by the vector $\beta(\omega,t)$. The equilibrium process consists of the following elements.

(i) A stochastic process for the vector of the J stock prices, $p_A(\omega,t) = \left[p_{A_j}(\omega,t)\right]_{j=1,\ldots,J}$, and the price of the bond, $p_B(\omega,t)$. All $J+1$ prices are continuous, square-integrable martingales with respect to the filtration $\{\mathcal{F}_t\}$.

(ii) A stochastic consumption price process $p_C(\omega,t)$. These are defined by

[10]For his existence theorem, Raimondo [32] imposes also three assumptions that are not included in the setting described above. Specifically, the utility functions $\varphi_i(\cdot) : i \in \{1,2\}$ are assumed to be bounded below: $\exists K > -\infty$ s.t. $\varphi_i(c) > K$ $\forall c \in \mathbb{R}_{++}$. Moreover, in order to not have to handle genericity considerations on existence, a short-sale constraint is introduced: $\exists M > 0$ s.t. the agent is not permitted to hold less than $-M$ units of any of the $J+1$ traded assets. Finally, the terminal-period endowment function is taken to satisfy $0 \leq \rho(\mathbf{x}) \leq r + e^{r|\mathbf{x}|}$ for some $r \in \mathbb{R}_+$ and $\forall \mathbf{x} \in \mathbb{R}^d$. Anderson and Raimondo [5] show that the first two assumptions are not necessary for existence of equilibrium in a model which nests the one examined here. As for the third condition, it is satisfied by any bounded-above function $\rho(\cdot)$. My results *per se* do not depend upon any assumptions other than the ones already stated in the text. Additional conditions, that may be necessary for an existence proof, are not really relevant for my comparative statics analysis. If an equilibrium price process does indeed exist, the equilibrium relative prices have to be as in (2) and this is where I begin.

$$
\begin{aligned}
p_{A_j}(\omega,t) &= \int_{\mathbb{R}^d}\cdots\int \varphi_2'\left(F(\omega,t,\mathbf{x})\right) e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)} \mathrm{d}\Phi(\mathbf{x}) \\
p_B(\omega,t) &= \int_{\mathbb{R}^d}\cdots\int \varphi_2'\left(F(\omega,t,\mathbf{x})\right) \mathrm{d}\Phi(\mathbf{x}) \\
p_C(\omega,t) &= \begin{cases} \varphi_1'(1) & t\in[0,T) \\ \varphi_1'\left(F(\omega,T,\mathbf{0})\right) & t=T \end{cases}
\end{aligned}
$$

where

$$
F(\omega,t,\mathbf{x}) = \rho\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right) + \sum_{i=1}^{J} e^{\mu_i T+\sigma_i\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)} \tag{1}
$$

is the terminal-period wealth (in units of consumption) and $\Phi(\cdot)$ is the cumulative distribution function for the standard d-dimensional normal.

This paper examines the comparative statics, with respect to changes in the Brownian vector $\beta(\omega,t): t\in[0,T)$, of the equilibrium relative price of the jth security:

$$
\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)} = \frac{\mathbb{E}_{\mathbf{x}}\left[\varphi_2'\left(F(\omega,t,\mathbf{x})\right) e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}\right]}{\mathbb{E}_{\mathbf{x}}\left[\varphi_2'\left(F(\omega,t,\mathbf{x})\right)\right]} \tag{2}
$$

where the expectations are taken with respect to $\mathbf{x}\sim\mathbf{N}(\mathbf{0},\mathbf{I}_d)$. I begin by deriving analytically the inner product of the row-vector σ_j with the gradient vector

$$
\nabla_{\beta(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right) = \left[\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)\right]_{k=1}^{d}
$$

of the equilibrium relative price of the jth stock (Theorem 2.1). This inner product is nonnegative; more precisely, it is strictly positive unless the jth terminal-period dividend does not depend upon any of the sources of uncertainty in the model (Corollary 2.1).

Theorem 2.1 *For any security $j \in \{1, ..., J\}$, we have*

$$\sum_{k=1}^{d} \sigma_{jk} \frac{\partial}{\partial \beta_k(\omega, t)} \left(\frac{p_{A_j}(\omega, t)}{p_B(\omega, t)} \right) = \frac{e^{\mu_j T + \sigma_j \beta(\omega,t)} \mathbb{E}_{(\mathbf{x},\mathbf{y})} \left[G(\mathbf{x}, \mathbf{y}) \sigma_j (\mathbf{y} - \mathbf{x}) e^{\sqrt{T-t}\sigma_j \mathbf{y}} \right]}{p_B(\omega, t)^2 \sqrt{(T-t)(2\pi)^{2d}}}$$

where $\mathbf{x}, \mathbf{y} \sim i.i.d.\mathbf{N}(\mathbf{0}, \mathbf{I}_d)$ *and* $G : \mathbb{R}^{2d} \to \mathbb{R}_{++}$ *is defined by*

$$G(\mathbf{x}, \mathbf{y}) = \varphi_2'(F(\omega, t, \mathbf{x})) \varphi_2'(F(\omega, t, \mathbf{y}))$$

Corollary 2.1 *For any security $j \in \{1, ..., J\}$, we have*

$$\sum_{k=1}^{d} \sigma_{jk} \frac{\partial}{\partial \beta_k(\omega, t)} \left(\frac{p_{A_j}(\omega, t)}{p_B(\omega, t)} \right) \geq 0 \quad \textit{with equality iff } \sigma_j' = \mathbf{0} \tag{3}$$

Clearly, the equilibrium relative price depends on the current realization, $\beta(\omega, t)$, of the underlying stochastic process in a way that exhibits generally rich and complex dynamics. Notice that the equilibrium price of the jth stock can be equivalently written as follows:

$$\begin{aligned} p_{A_j}(\omega, t) &= \text{Cov}_{\mathbf{x}} \left[\varphi_2'(F(\omega, t, \mathbf{x})), e^{\mu_j T + \sigma_j \left(\beta(\omega,t) + \sqrt{T-t}\mathbf{x} \right)} \right] \\ &\quad + \mathbb{E}_{\mathbf{x}} \left[\varphi_2'(F(\omega, t, \mathbf{x})) \right] \mathbb{E}_{\mathbf{x}} \left[e^{\mu_j T + \sigma_j \left(\beta(\omega,t) + \sqrt{T-t}\mathbf{x} \right)} \right] \\ &= \text{Cov}_{\mathbf{x}} \left[\varphi_2'(F(\omega, t, \mathbf{x})), e^{\mu_j T + \sigma_j \left(\beta(\omega,t) + \sqrt{T-t}\mathbf{x} \right)} \right] + p_B(\omega, t) e^{\mu_j T + \sigma_j \left(\beta(\omega,t) + \frac{(T-t)}{2} \sigma_j' \right)} \end{aligned}$$

Its partial derivative, with respect to the kth component of the Brownian vector, is given by:

$$\begin{aligned} \frac{\partial p_{A_j}(\omega, t)}{\partial \beta_k(\omega, t)} &= \frac{\partial}{\partial \beta_k(\omega, t)} \text{Cov}_{\mathbf{x}} \left[\varphi_2'(F(\omega, t, \mathbf{x})), e^{\mu_j T + \sigma_j \left(\beta(\omega,t) + \sqrt{T-t}\mathbf{x} \right)} \right] \\ &\quad + e^{\mu_j T + \sigma_j \left(\beta(\omega,t) + \frac{(T-t)}{2} \sigma_j' \right)} \frac{\partial p_B(\omega, t)}{\partial \beta_k(\omega, t)} \\ &\quad + \sigma_{jk} p_B(\omega, t) e^{\mu_j T + \sigma_j \left(\beta(\omega,t) + \frac{(T-t)}{2} \sigma_j' \right)} \end{aligned} \tag{4}$$

For the partial derivative of the equilibrium price of the bond and the equilibrium relative price of the jth stock, respectively, we have

$$\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)} = e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\frac{(T-t)}{2}\sigma_j'\right)} \frac{\mathbb{E}_{\mathbf{x}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{x}+\sqrt{T-t}\sigma_j'\right)\right)\right]}{\mathbb{E}_{\mathbf{x}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)\right]} \tag{5}$$

$$\frac{\partial p_B(\omega,t)}{\partial \beta_k(\omega,t)} = \mathbb{E}_{\mathbf{x}}\left[\varphi_2''\left(F\left(\omega,t,\mathbf{x}\right)\right)\frac{\partial F(\omega,t,\mathbf{x})}{\partial \beta_k(\omega,t)}\right] \tag{6}$$

Given the current realization $\beta(\omega,t)$, exchanging at time t one unit of the bond for one unit of the jth stock increases the expected terminal-period wealth by the expected jth terminal-period dividend $e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\frac{(T-t)}{2}\sigma_j'\right)}$. In terms of terminal-period wealth, therefore, one unit of the jth stock is equivalent to $e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\frac{(T-t)}{2}\sigma_j'\right)}$ units of the bond. In expected marginal utility terms, however, the equivalence relation requires also that any future realization $\sqrt{T-t}\mathbf{x} \sim \mathbf{N}\left(0,(T-t)\mathbf{I}_d\right)$ of the stochastic process $\beta(\omega,T)-\beta(\omega,t)$ is translated by $(T-t)\sigma_j'$. Equation (5) depicts this.

The Own-Dividend Effect

Other things remaining equal, a change in the kth component, $\mathrm{d}\beta_k(\omega,t)$, of the stochastic vector-process $\beta(\omega,t)$ alters the $\mathcal{F}_t$-conditional drift, $\mu_j T+\sigma_j\beta(\omega,t)$, of the underlying stochastic process that determines the terminal-period dividend of the jth stock.[11] The $\mathcal{F}_t$-conditional expected terminal-period dividend of the jth risky security changes by $\sigma_{jk}e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\frac{(T-t)}{2}\sigma_j'\right)}\mathrm{d}\beta_k(\omega,t)$. Suppose that $\beta_k(\omega,t)$ increases. If $\sigma_{jk}>0$ ($\sigma_{jk}<0$), the expected terminal-period dividend will now be higher (lower). Due to non-satiation ($\varphi_2'(\cdot)>0$), this increases (decreases) the willingness of the agent to hold the jth stock and, since she must hold the net supply of each asset in equilibrium, $p_{A_j}(\omega,t)$ must rise (fall) exactly by $p_B(\omega,t)\,\sigma_{jk}e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\frac{(T-t)}{2}\sigma_j'\right)}\mathrm{d}\beta_k(\omega,t)$. This is the *own-dividend effect* of the change in $\beta_k(\omega,t)$ on the equilibrium price of the jth stock. It is depicted by the third term on the right-hand side of equation (4).

[11] "Other things remaining unchanged" (or similar expressions) refer henceforth to the current realizations of the remaining $d-1$ sources of uncertainty, $\beta_m(\omega,t) : m \in \{1,...,d\}\setminus\{k\}$.

The Wealth Effect

For any future realization $\sqrt{T-t}\mathbf{x}$ of the stochastic process $\beta(\omega,T)-\beta(\omega,t)$, a change in $\beta_k(\omega,t)$ corresponds to revealing information that changes also the terminal-period dividend of every stock $i \in \{1,...,J\}$ by $\sigma_{ik}e^{\mu_i T+\sigma_i\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}\mathrm{d}\beta_k(\omega,t)$. The changes in the J dividends along with that in the terminal-period endowment, $\mathrm{d}\rho\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)$, give the corresponding change in terminal-period wealth. Other things being equal, due to the agent's risk aversion ($\varphi_2''(\cdot)<0$), this induces an opposite change in marginal utility; I will call this the *wealth effect* of $\mathrm{d}\beta_k(\omega,t)$. The wealth effect on the equilibrium price of the bond is given by equation (6). That on the equilibrium price of the jth stock is given by the second term on the right-hand side of equation (4) (in terms of terminal-period wealth, one unit of the stock is equivalent to $e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\frac{(T-t)}{2}\sigma_j'\right)}$ units of the bond).

The Asset-Riskiness Effect

Recall equation (2). For any future realization $\sqrt{T-t}\mathbf{x}$, the extent to which $\mathrm{d}\beta_k(\omega,t)$ alters the equilibrium price $p_{A_j}(\omega,t)$ through a change in the marginal utility of terminal-period wealth $\varphi_2'(F(\omega,t,\mathbf{x}))$ depends on the realization of the terminal-period dividend $e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}$. Similarly, the extent to which $\mathrm{d}\beta_k(\omega,t)$ alters $p_{A_j}(\omega,t)$ through a change in the jth terminal-period dividend depends on the realization of the marginal utility of terminal-period wealth. That is, $\mathrm{d}\beta_k(\omega,t)$ affects $p_{A_j}(\omega,t)$ also via changes in the correlation between the marginal utility of terminal-period wealth and the terminal-period dividend of the jth risky security. I will be referring to this as the *asset-riskiness effect* of $\mathrm{d}\beta_k(\omega,t)$ on the equilibrium price of the jth stock. It is depicted by the first term on the right-hand side of equation (4).

To understand the mechanism of the asset-riskiness effect, it is instructive to consider a setting in which the components of the Brownian process that determine the jth terminal-period dividend ($\beta_m(\omega,t) : \sigma_{jm}\neq 0$) affect the terminal-period wealth only through this dividend. Specifically, let $\mathbf{z}\sim N(\mathbf{0},\mathbf{I}_M)$ denote the array of x_m's with $\sigma_{jm}\neq 0$, for some positive integer $M < d$. Let also $\mathbf{y} \sim N\left(\mathbf{0}, \mathbf{I}_{d-M}\right)$ and $\mathbf{x} = (\mathbf{y}, \mathbf{z})$. Moreover, suppose that the terminal-period wealth is given by

$$F\left(\omega, t, \mathbf{x}\right) = \rho\left(\omega, t, \mathbf{y}\right) + \sum_{i \neq j} e^{\mu_i T + \sigma_i \left(\beta(\omega,t) + \sqrt{T-t}\mathbf{y}\right)} + e^{\mu_j T + \sigma_j \left(\beta(\omega,t) + \sqrt{T-t}\mathbf{z}\right)} \tag{7}$$

for some continuous function $\rho : \mathbb{R}^{d-M} \to \mathbb{R}_+$.[12] In this case, the asset-riskiness effect is given by

$$\begin{aligned}
&\mathrm{Cov}_{\mathbf{x}}\left[\varphi_2''\left(F\left(\omega,t,\mathbf{x}\right)\right)\frac{\partial F\left(\omega,t,\mathbf{y}\right)}{\partial \beta_k\left(\omega,t\right)}, e^{\mu_j T + \sigma_j \left(\beta(\omega,t) + \sqrt{T-t}\mathbf{z}\right)}\right] \\
= &\int_{\mathbf{y}} \begin{pmatrix} \int_{\mathbf{z}} \varphi_2''\left(F\left(\omega,t,\mathbf{x}\right)\right) e^{\mu_j T + \sigma_j \left(\beta(\omega,t) + \sqrt{T-t}\mathbf{z}\right)} \mathrm{d}\Phi\left(\mathbf{z}\right) - \\ \int_{\mathbf{z}} \varphi_2''\left(F\left(\omega,t,\mathbf{x}\right)\right)\mathrm{d}\Phi\left(\mathbf{z}\right) \int_{\mathbf{z}} e^{\mu_j T + \sigma_j \left(\beta(\omega,t) + \sqrt{T-t}\mathbf{z}\right)} \mathrm{d}\Phi\left(\mathbf{z}\right) \end{pmatrix} \frac{\partial F\left(\omega,t,\mathbf{y}\right)}{\partial \beta_k\left(\omega,t\right)} \mathrm{d}\Phi\left(\mathbf{y}\right) \\
= &\int_{\mathbf{y}} \mathrm{Cov}_{\mathbf{z}}\left[\varphi_2''\left(F\left(\omega,t,\mathbf{x}\right)\right), e^{\mu_j T + \sigma_j \left(\beta(\omega,t) + \sqrt{T-t}\mathbf{z}\right)}\right] \frac{\partial F\left(\omega,t,\mathbf{y}\right)}{\partial \beta_k\left(\omega,t\right)} \mathrm{d}\Phi\left(\mathbf{y}\right)
\end{aligned}$$

By (7), conditional on $\mathbf{y}$, the realization $F\left(\omega, t, \mathbf{x}\right)$ is strictly increasing in the realization $e^{\mu_j T + \sigma_j \left(\beta(\omega,t) + \sqrt{T-t}\mathbf{z}\right)}$. Under non-increasing risk-aversion, the two are comonotonic and the covariance in the integrand of the last integral above is strictly positive (see Appendix B). The sign of the asset-riskiness effect is the same as that of the derivative $\frac{\partial F(\omega,t,\mathbf{y})}{\partial \beta_k(\omega,t)}$.

The intuition for why, in this case, the asset-riskiness effect of $\mathrm{d}\beta_k\left(\omega, t\right)$ on the equilibrium price $p_{A_j}\left(\omega, t\right)$ reinforces its wealth effect is straightforward. Let $\frac{\partial F(\omega,t,\mathbf{y})}{\partial \beta_k(\omega,t)} > 0$ so that an increase in $\beta_k\left(\omega, t\right)$ raises the $\mathcal{F}_t$-conditional terminal period wealth and, by risk aversion, reduces its marginal utility $\varphi_2'\left(F\left(\omega, t, \mathbf{x}\right)\right)$. Under non-increasing absolute risk-aversion, the decrease in $\varphi_2'\left(F\left(\omega, t, \mathbf{x}\right)\right)$ is smaller when the jth terminal-period dividend is large and larger when $A_j\left(\omega, T\right)$ is small. That is, the increase in $\beta_k\left(\omega, t\right)$ makes the marginal utility of terminal-period wealth and the terminal-period dividend of the jth security less negatively correlated. Equivalently, the terminal-period wealth of the agent and $A_j\left(\omega, T\right)$ become less positively correlated. The perceived "riskiness" of the jth security decreases, increasing

[12] For an example of such a setting, see Proposition 4.1.5.

$p_{A_j}(\omega, t)$.

Once we depart from the terminal-period wealth specification in (7), however, the dynamics of the asset-riskiness effect become hard to discern. Resulting from $\mathrm{d}\beta_k(\omega, t)$, the new level of terminal-period wealth is $F(\omega, t, \mathbf{x}) + \mathrm{d}F(\omega, t, \mathbf{x})$. The new covariance of the marginal utility of terminal-period wealth with the jth terminal-period dividend is given by

$$\begin{aligned} & \mathrm{Cov}_{\mathbf{x}}\left[\varphi_2'\left(F(\omega, t, \mathbf{x}) + \mathrm{d}F(\omega, t, \mathbf{x})\right), e^{\mu_j T + \sigma_j\left(\beta(\omega,t) + \mathrm{d}\beta_k(\omega,t)\mathbf{e}_k + \sqrt{T-t}\mathbf{x}\right)}\right] \\ = \; & e^{\sigma_{jk}\mathrm{d}\beta_k(\omega,t)}\mathrm{Cov}_{\mathbf{x}}\left[\varphi_2'\left(F(\omega, t, \mathbf{x}) + \mathrm{d}F(\omega, t, \mathbf{x})\right), e^{\mu_j T + \sigma_j\left(\beta(\omega,t) + \sqrt{T-t}\mathbf{x}\right)}\right] \end{aligned}$$

Let again $\frac{\partial F(\omega,t,\mathbf{x})}{\partial \beta_k(\omega,t)} > 0$ and suppose that the comonotonicity argument applies so that, under non-increasing risk-aversion, the covariance on the right-hand side above is less negative than the initial covariance, $\mathrm{Cov}_{\mathbf{x}}\left[\varphi_2'\left(F(\omega, t, \mathbf{x})\right), e^{\mu_j T + \sigma_j\left(\beta(\omega,t) + \sqrt{T-t}\mathbf{x}\right)}\right]$. Assume, though, $\sigma_{jk}\mathrm{d}\beta_k(\omega, t) > 0$. Since $\mathrm{sign}\left\{e^{\sigma_{jk}\mathrm{d}\beta_k(\omega,t)} - 1\right\} \equiv \mathrm{sign}\left\{\sigma_{jk}\mathrm{d}\beta_k(\omega, t)\right\}$, the increase in the jth terminal-period dividend could be sufficient to make the covariance on the left-hand side above more negative overall. That is, the perceived "riskiness" of the jth security could actually increase, decreasing $p_{A_j}(\omega, t)$.

Notice that the preceding paragraph sketched a scenario where the asset-riskiness effect of $\mathrm{d}\beta_k(\omega, t)$ on the equilibrium price $p_{A_j}(\omega, t)$ opposes its wealth effect. If the comonotonicity argument does not apply, this can occur even in very simple settings with $\sigma_{jk} = 0$ (see Proposition 4.1.3).

Combining the Three Effects

The main objective of my analysis is to sign the partial derivative

$$\frac{\partial}{\partial \beta_k(\omega, t)}\left(\frac{p_{A_j}(\omega, t)}{p_B(\omega, t)}\right) = \frac{1}{p_B(\omega, t)}\left[\frac{\partial p_{A_j}(\omega, t)}{\partial \beta_k(\omega, t)} - \left(\frac{p_{A_j}(\omega, t)}{p_B(\omega, t)}\right)\left(\frac{\partial p_B(\omega, t)}{\partial \beta_k(\omega, t)}\right)\right] \tag{8}$$

Since

$$\operatorname{sign}\left\{\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)\right\}\equiv\operatorname{sign}\left\{\frac{\frac{\partial p_{A_j}(\omega,t)}{\partial\beta_k(\omega,t)}}{p_{A_j}(\omega,t)}-\frac{\frac{\partial p_B(\omega,t)}{\partial\beta_k(\omega,t)}}{p_B(\omega,t)}\right\} \tag{9}$$

the monotonicity of the relation between the kth component of the Brownian vector, $\beta_k(\omega,t)$, and the equilibrium relative price $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ depends on the difference in the relative changes of $p_{A_j}(\omega,t)$ and $p_B(\omega,t)$. The resulting dynamics are generally complex to an extent that they are impossible to predict using only economic intuition. First, the wealth effects on $p_{A_j}(\omega,t)$ and $p_B(\omega,t)$ pull $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ in opposite directions. Moreover, the own-dividend effect pushes $p_{A_j}(\omega,t)$ always in the opposite direction than its wealth effect. Finally, if the terminal-period utility $\varphi_2(\cdot)$ exhibits non-increasing absolute risk aversion, the asset-riskiness effect may pull $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ in the opposite direction than the combination of the wealth effects on $p_{A_j}(\omega,t)$ and $p_B(\omega,t)$.

Theorem 2.1 addresses this for $\sum_{k=1}^{d}\sigma_{jk}\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)$, the summation of the rates of change of the equilibrium relative price of the jth risky security (with respect to the d components of the Brownian vector) that has as coefficients the corresponding factor loadings. In terms of relative changes in absolute prices, summing over the d dimensions of the Brownian vector and using as coefficients the factor loadings of the security, the sum of the relative changes in the equilibrium price of the stock outweighs the sum of the relative changes in the equilibrium price of the bond. This holds always, as long as the terminal-period dividend of the security depends on the realization of at least one source of uncertainty.[13]

[13] For $\sigma_j' = \mathbf{0}$, equation (2) gives $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)} = e^{\mu_j T}$. The equilibrium relative price is a constant, independent of the realizations of the vector $\beta(\omega,t)$: $\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right) = 0$, $k\in\{1,...,d\}$. Take any $k\in\{1,...,d\}$. Since $\sigma_{jk}=0$, $\beta_k(\omega,t)$ does not affect the terminal-period dividend $A_j(\omega,T)$; there is no own-dividend effect on the equilibrium price $p_{A_j}(\omega,t)$. Moreover, since all other factor loadings of the jth stock are also zero ($\sigma_{jk'}=0 : k'\in\{1,...,d\}\setminus\{k\}$), $A_j(\omega,T)$ is independent of the subsequent path $\{\beta(\omega,\prime):\prime\in(t,T]\}$ of the Brownian process and, consequently, of the terminal-period wealth. Clearly, a change in $\beta_k(\omega,t)$ induces no asset-riskiness effect on $p_{A_j}(\omega,t)$ while the wealth effects on the equilibrium prices $p_{A_j}(\omega,t)$ and $p_B(\omega,t)$

The intuition behind the theorem is straightforward when

(i) the terminal-period dividend of the jth security varies with the realization of only one source of uncertainty, $\beta_m(\omega,t)$ for $m\in\{1,...,d\}$, $A_j(\omega,T)=e^{\mu_jT+\sigma_{jm}\left(\beta_m(\omega,t)+\sqrt{T-t}x_m\right)}$, and

(ii) this source of uncertainty affects the terminal-period wealth only through the jth dividend: $\frac{\partial\rho(\cdot)}{\partial\beta_m(\omega,T)}=0\ \forall\omega\in\Omega$ and $\sigma_{im}=0\ \forall i\in\{1,...,J\}\setminus\{j\}$.

Let the mth component of the Brownian process change from $\beta_m(\omega,t)$ to $\beta_m(\omega,t)+\mathrm{d}\beta_m(\omega,t)$. The terminal-period wealth changes only through the jth terminal-period dividend which is now given by

$$e^{\mu_jT+\sigma_{jm}\left(\beta_m(\omega,t)+\mathrm{d}\beta_m(\omega,t)+\sqrt{T-t}x_m\right)}=e^{\sigma_{jm}\mathrm{d}\beta_m(\omega,t)}A_j(\omega,T)$$

Since the agent is everywhere non-satiated ($\varphi_2'(\cdot)>0$) and the terminal-period endowment is unaffected by $\mathrm{d}\beta_m(\omega,t)$, her preferences for the risky asset change in the direction of First-order Stochastic Dominance (FSD). Suppose that $\beta_m(\omega,t)$ increases (decreases). For $\sigma_{jm}>0$, the new terminal-period dividend dominates (is dominated by) the old in the sense of FSD. The agent is now more (less) willing to hold the stock. Moreover, the wealth effect on the equilibrium price of the bond is negative (positive). For $\sigma_{jm}<0$, the old terminal-period dividend dominates (is dominated by) the new in terms of FSD whereas the wealth effect on $p_B(\omega,t)$ is positive (negative). In either case, $\sigma_{jm}\frac{\partial}{\partial\beta_m(\omega,t)}\left(\frac{p_A(\omega,t)}{p_B(\omega,t)}\right)>0$.[14]

In more general settings, Theorem 2.1 can be viewed as generalizing this argument. The proof of Section 5 uses straightforward mathematical apparatus but is quite subtle in

cancel each other out.

[14] Put differently, when $\sigma_{jm}>0$ ($\sigma_{jm}<0$), going from the old to the new terminal-period dividend is in the opposite (same) direction as Proposition 1 in Gollier [22], the factor being $e^{\sigma_{jm}\mathrm{d}\beta_m(\omega,t)}$. For any risk-averse individual, $\mathrm{d}\beta_m(\omega,t)$ increases (reduces) the optimal demand and, consequently, the equilibrium relative price of the jth risky security. Of course, Gollier [22] studies probability distributions whose supports are closed intervals but his results are straightforward to generalize in the current setting (see Lemma A.1 in the Appendix).

its reasoning, especially with respect to its last (and most crucial) step. It attests to the complexity of the dynamics of the relation between the $\mathcal{F}_t$-conditional equilibrium relative price and the current realization $\beta(\omega,t)$ of the underlying vector stochastic process. The next two sections examine settings with sufficient structure for stronger conclusions to be made.

3 One Source of Uncertainty ($d = 1$)

When $\beta(\omega,t)$ and, consequently, σ_j are scalars, Corollary 2.1 can be stated as follows.

Corollary 3.1 *Suppose that there is a single source of uncertainty in the economy, represented by the (scalar) stochastic process $\beta(\omega,t)$. Then, for any security $j \in \{1,...,J\}$, the equilibrium relative price process $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ depends on $\beta(\omega,t)$ if and only if $\sigma_j \neq 0$. Specifically, it is strictly increasing (strictly decreasing) in $\beta(\omega,t)$ if $\sigma_j > 0$ ($\sigma_j < 0$).*

With only one source of uncertainty, the derivative of the equilibrium relative price with respect to the current realization of the (scalar) Brownian motion has always the same sign as the factor loading of the stock. In particular, when $\beta(\omega,t)$ increases, $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ increases (decreases) if the jth terminal dividend is positively (negatively) correlated with $\beta(\omega,t)$. As long as the terminal-period dividend of a security depends on $\beta(\omega,t)$, the path of its equilibrium relative price process identifies uniquely the path on which the underlying uncertainty is being resolved.

For example, let $\frac{\partial F(\omega,t,\cdot)}{\partial\beta(\omega,t)} > 0$ and the agent exhibit DARA. If $\sigma_j > 0$, an increase in $\beta(\omega,t)$ raises the $\mathcal{F}_t$-conditional terminal-period dividend of the jth stock, pushing $p_{A_j}(\omega,t)$ upwards through the own-dividend effect. Since the terminal-period wealth is now higher, the DARA property dictates that the combination of the wealth effects on $p_B(\omega,t)$ and $p_{A_j}(\omega,t)$ pushes $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ also upwards. However, the asset-riskiness effect on $p_{A_j}(\omega,t)$ may pull it downwards (footnote 16). The corollary guarantees that the combination of the two upward forces dominate the downward one so that, while the equilibrium price of the bond necessarily falls, the equilibrium price of the risky security either increases or decreases by less in percentage terms.

4 Multiple Sources of Uncertainty ($d > 1$)

With more than one sources of uncertainty, the implications of Corollary 2.1 are subtle even when $J = 1$. It is instructive to distinguish between whether the terminal-period dividend $A_j(\omega, T)$ depends upon a single or multiple sources of uncertainty.

4.1 One Source of Uncertainty in the Dividend

Suppose first that $A_j(\omega, T)$ varies with only one source of uncertainty, $\beta_m(\omega, T)$, for some $m \in \{1, ..., d\}$. That is,

$$\sigma_j = \sigma_{jm}\mathbf{e}'_m \qquad \sigma_{jm} \in \mathbb{R} \quad m \in \{1, ..., d\}$$

where $\mathbf{e}_m \in \mathbb{R}^d$ denotes the vector with 1 at its mth entry and zeroes elsewhere.

4.1.1 Monotonicity in Own Dividend

Since $\sigma_{jk} = 0$ for $k \in \{1, ..., d\} \setminus \{m\}$, Corollary 2.1 reads now

Corollary 4.1 *For $j \in \{1, ..., J\}$, let the terminal-period dividend $A_j(\omega, T)$ depend on only one source of uncertainty, $\beta_m(\omega, T)$ for $m \in \{1, ..., d\}$. The equilibrium relative price process $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ is monotone in $\beta_m(\omega, t)$; specifically, it is strictly increasing (strictly decreasing) if $\sigma_{jm} > 0$ ($\sigma_{jm} < 0$).*

Notice that the corollary applies to every security $j \in \{1, ..., J\}$ in the model when the matrix σ of factor loadings is diagonal:

$$J = d \qquad \sigma = [\sigma_{11}\mathbf{e}_1, ..., \sigma_{dd}\mathbf{e}_d] \qquad \sigma_{kk} \in \mathbb{R},\ k \in \{1, ..., d\}$$

4.1.2 Cross-correlations

When $\sigma_{jk} = 0$, (3) does not provide any information towards the comparative statics of $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ with $\beta_k(\omega,t)$. Yet, apart from quite special cases, the equilibrium relative price process does vary with $\beta_k(\omega,t)$ in a way that depends on the actual functional forms of the terminal-period wealth $F(\cdot)$ and utility $\varphi_2(\cdot)$.

Since $\sigma_{jk} = 0$, a change in $\beta_k(\omega,t)$ does not affect the $\mathcal{F}_t$-conditional drift, $\mu_j T + \sigma_j \beta(\omega,t)$, of the stochastic process underlying the dividend $A_j(\omega,T)$. That is, there is no own-dividend effect, only wealth and asset-riskiness effects. To facilitate the mathematical analysis and illustrate the importance of the asset-riskiness effect for the richness of the equilibrium relative price dynamics, I will consider first

(i) CARA terminal-period utility: $\varphi_2(c) = \gamma e^{\alpha c} \quad \gamma, \alpha < 0$. This ensures that the combination of the wealth effects of $\mathrm{d}\beta_k(\omega,t)$ on the absolute prices $p_{A_j}(\omega,t)$ and $p_B(\omega,t)$ leaves the relative price $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ unaffected.

(ii) Specifications for the terminal-period wealth $F(\cdot)$ that allow the effects of $\mathrm{d}\beta_k(\omega,T)$ and $\mathrm{d}\beta_m(\omega,T)$ to be separate.

With respect to (ii) above, suppose first that the following conditions hold.

(A.i) The kth component of the Brownian vector affects only the ith ($i \in \{1,...,J\} \setminus \{j\}$) terminal-period dividend $A_i(\omega,T)$ which, in turn, depends only on $\beta_k(\omega,T)$. That is, the kth column and ith row of the matrix σ are given by, respectively, $\sigma_{ik}\mathbf{e}_i$ and $\sigma_i = \sigma_{ik}\mathbf{e}'_k$.

(A.ii) The terminal-period endowment process $\rho(\cdot)$ is given by

$$\rho(\beta(\omega,T)) = \rho_{-k}(\beta_{-k}(\omega,T)) + \rho_k(\beta_k(\omega,T)) \qquad \forall \omega \in \Omega$$

for some continuous functions $\rho_{-k} : \mathbb{R}^{d-1} \to \mathbb{R}_+$ and $\rho_k : \mathbb{R} \to \mathbb{R}_+$.

The terminal-period wealth in (1) can be expressed now as follows

$$\begin{aligned}
F(\omega,t,\mathbf{x}) &= \rho_{-k}\left(\left(\sum_{l\neq k,l=1}^{d}\mathbf{e}_l'\right)\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)\right)+\rho_k\left(\beta_k(\omega,t)+\sqrt{T-t}x_k\right) \\
&\quad+\sum_{h\neq i,h=1}^{J} e^{\mu_h T+\sigma_h\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}+e^{\mu_i T+\sigma_{ik}\left(\beta_k(\omega,t)+\sqrt{T-t}x_k\right)} \\
&= F_{-k}(\omega,t,\mathbf{x}_{-k})+F_k(\omega,t,x_k) \qquad (10)
\end{aligned}$$

Under the specification in (10) and CARA terminal-period utility, the kth component of the Brownian vector does not affect the equilibrium relative price process of the jth security.

Proposition 4.1.1 *For $j\in\{1,...,J\}$, let its terminal-period dividend depend on the realization of only the mth source of uncertainty: $A_j(\omega,t)=e^{\mu_j T+\sigma_{jm}\left(\beta_m(\omega,t)+\sqrt{T-t}x_m\right)}$. Suppose also that*

(i) the terminal-period wealth process is specified as in (10), and

(ii) the terminal-period utility $\varphi_2(\cdot)$ exhibits CARA.

Then, for $k\neq m$, $\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)=0$

With CARA terminal-period utility, changes in $\beta_k(\omega,t)$ do not matter for the equilibrium relative price of the jth security also in another setting. When the stochastic process $\beta_m(\omega,T)$ that determines the dividend $A_j(\omega,T)$ affects the agent's terminal-period wealth in a separate way from the remaining $d-1$ components of the Brownian vector. Specifically, for some continuous functions $\rho_{-m}:\mathbb{R}^{d-1}\to\mathbb{R}_+$ and $\rho_m:\mathbb{R}\to\mathbb{R}_+$, let

$$\rho(\beta(\omega,T))=\rho_{-m}(\beta_{-m}(\omega,T))+\rho_m(\beta_m(\omega,T)) \qquad \forall\omega\in\Omega$$

so that the terminal-period wealth is given by

$$\begin{aligned} F(\omega,t,\mathbf{x}) &= \rho_{-m}\left(\sum_{l\neq m,l=1}^{d}\mathbf{e}_l'\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)\right)+\rho_m\left(\beta_m(\omega,t)+\sqrt{T-t}x_m\right) \\ &\quad +\sum_{h\neq j,h=1}^{J}e^{\mu_h T+\sigma_h\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}+e^{\mu_j T+\sigma_{jm}\left(\beta_k(\omega,t)+\sqrt{T-t}x_m\right)} \\ &= F_{-m}(\omega,t,\mathbf{x}_{-m})+F_m(\omega,t,x_m) \end{aligned} \tag{11}$$

Proposition 4.1.2 *For $j\in\{1,...,J\}$, let $A_j(\omega,T)=e^{\mu_j T+\sigma_{jm}\left(\beta_m(\omega,t)+\sqrt{T-t}x_m\right)}$. Suppose also that*

(i) the terminal-period wealth process is specified as in (11), and

(ii) the terminal-period utility function $\varphi_2(\cdot)$ exhibits CARA.

Then, for $k\neq m$, $\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)=0$

Observe that a special case of the specification in either (10) or (11) obtains when the matrix of factor loadings σ is diagonal and, for some continuous functions $\rho_i:\mathbb{R}\to\mathbb{R}_+$, we have

$$\rho(\beta(\omega,T))=\sum_{i=1}^{d}\rho_i\left(\beta_i(\omega,t)+\sqrt{T-t}x_i\right)$$

The terminal-period wealth can now be written as

$$\begin{aligned} F(\omega,t,\mathbf{x}) &= \sum_{i=1}^{d}\rho_i\left(\beta_i(\omega,t)+\sqrt{T-t}x_i\right)+\sum_{i=1}^{d}e^{\mu_i T+\sigma_i\left(\beta_i(\omega,t)+\sqrt{T-t}x_i\right)} \\ &= \sum_{i=1}^{d}F_i(\omega,t,x_i) \end{aligned} \tag{12}$$

and it can be verified analytically that, under the conditions of either of Propositions 4.1.1 and 4.1.2, the equilibrium relative price $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ is not a function of $\beta_k(\omega,t)$. In the case of the terminal-wealth specification in (10), equation (2) gives

$$\begin{aligned}\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)} &= e^{\mu_m T+\sigma_{jm}\beta_m(\omega,t)+\frac{(T-t)\sigma_{jm}^2}{2}} \\ &\quad \frac{\mathbb{E}_{x_k}\left[e^{\alpha F_k(\omega,t,x_k)}\right]}{\mathbb{E}_{x_k}\left[e^{\alpha F_k(\omega,t,x_k)}\right]}\frac{\prod_{i\neq k}\mathbb{E}_{\mathbf{x}_{-k}}\left[e^{\alpha F_{-k}\left(\omega,t,\mathbf{x}_{-k}+\sqrt{T-t}\sigma_{jm}\mathbf{e}_{-k}\right)}\right]}{\prod_{i\neq k}\mathbb{E}_{\mathbf{x}_{-k}}\left[e^{\alpha F_{-k}(\omega,t,\mathbf{x}_{-k})}\right]} \\ &= e^{\mu_m T+\sigma_{jm}\beta_m(\omega,t)+\frac{(T-t)\sigma_{jm}^2}{2}}\frac{\prod_{i\neq k}\mathbb{E}_{\mathbf{x}_{-k}}\left[e^{\alpha F_{-k}\left(\omega,t,\mathbf{x}_{-k}+\sqrt{T-t}\sigma_{jm}\mathbf{e}_{-k}\right)}\right]}{\prod_{i\neq k}\mathbb{E}_{\mathbf{x}_{-k}}\left[e^{\alpha F_{-k}(\omega,t,\mathbf{x}_{-k})}\right]}\end{aligned}$$

Under the specification in (11),

$$\begin{aligned}\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)} &= e^{\mu_m T+\sigma_{jm}\beta_m(\omega,t)+\frac{(T-t)\sigma_{jm}^2}{2}} \\ &\quad \frac{\mathbb{E}_{x_m}\left[e^{\alpha F_m\left(\omega,t,x_m+\sqrt{T-t}\sigma_{jm}\right)}\right]}{\mathbb{E}_{x_m}\left[e^{\alpha F_m(\omega,t,x_m)}\right]}\frac{\prod_{i\neq m}\mathbb{E}_{x_i}\left[e^{\alpha F_i(\omega,t,x_i)}\right]}{\prod_{i\neq m}\mathbb{E}_{x_i}\left[e^{\alpha F_i(\omega,t,x_i)}\right]} \\ &= e^{\mu_m T+\sigma_{jm}\beta_m(\omega,t)+\frac{(T-t)\sigma_{jm}^2}{2}}\frac{\mathbb{E}_{x_m}\left[e^{\alpha F_m\left(\omega,t,x_m+\sqrt{T-t}\sigma_{jm}\right)}\right]}{\mathbb{E}_{x_m}\left[e^{\alpha F_m(\omega,t,x_m)}\right]}\end{aligned}$$

while, for the specification in (12), we set $m=j$ and obtain

$$\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)} = e^{\mu_j T+\sigma_{jj}\beta_j(\omega,t)+\frac{(T-t)\sigma_{jj}^2}{2}}\frac{\mathbb{E}_{x_j}\left[e^{\alpha F_j\left(\omega,t,x_j+\sigma_{jj}\sqrt{T-t}\right)}\right]}{\mathbb{E}_{x_j}\left[e^{\alpha F_j(\omega,t,x_j)}\right]}$$

Example 4.1.1 *Let the terminal-period wealth be specified as in (12) and $\varphi_2(\cdot)$ exhibit CARA. By Theorem 2.1 and either of Propositions 4.1.1 and 4.1.2, we have*

$$\frac{\partial}{\partial\beta_k}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right) = \begin{cases}\frac{e^{\mu_j T+\sigma_{jj}\beta_j(\omega,t)}}{p_B(\omega,t)^2}\mathbb{E}_{(\mathbf{x},\mathbf{y})}\left[G(\mathbf{x},\mathbf{y})\,\sigma_{jj}(y_j-x_j)\,e^{\sqrt{T-t}\sigma_{jj}y_j}\right] & \text{if } k=j \\ 0 & \text{if } k\neq j\end{cases}$$

The jth row of the Jacobian matrix of equilibrium relative prices is given by

$$\mathbf{J}(\omega,t)_j = \sigma_{jj}\frac{e^{\mu_j T+\sigma_{jj}\beta_j(\omega,t)}}{p_B(\omega,t)^2}\mathbb{E}_{(\mathbf{x},\mathbf{y})}\left[G(\mathbf{x},\mathbf{y})(y_j-x_j)\,e^{\sqrt{T-t}\sigma_{jj}y_j}\right]\mathbf{e}_j'$$

That is, $\mathbf{J}(\omega, t)$ is diagonal, its jth diagonal entry having the sign of the corresponding factor loading σ_{jj}. The Jacobian matrix of equilibrium relative prices is non-singular.

The Insufficiency of CARA for Zero Cross-correlations

Propositions 4.1.1 and 4.1.2 agree with the standard intuition that, under CARA, the wealth effects on the equilibrium prices $p_{A_j}(\omega, t)$ and $p_B(\omega, t)$ cancel each other out and, therefore, do not matter for the equilibrium relative price. The stringency of their respective conditions indicate, however, that neutralizing the wealth effects is not sufficient for $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ to not respond to changes in $\beta_k(\omega, t)$ when $\sigma_{jk} = 0$.

In Section 5, I establish that the rates of equilibrium price changes can be also expressed as follows (recall that $\sqrt{T-t}x_k \sim \mathbf{N}(0, T-t)$)

$$\begin{aligned}
\frac{\partial p_{A_j}(\omega, t)}{\partial \beta_k(\omega, t)} &= \frac{1}{T-t}\mathbb{E}_{\mathbf{x}}\left[\sqrt{T-t}x_k\varphi_2'(F(\omega, t, \mathbf{x}))\, e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}\right] \\
&= \frac{1}{T-t}\mathrm{Cov}_{\mathbf{x}}\left[\sqrt{T-t}x_k,\ \varphi_2'(F(\omega, t, \mathbf{x}))\, e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}\right] \\
\frac{\partial p_B(\omega, t)}{\partial \beta_k(\omega, t)} &= \mathbb{E}_{\mathbf{x}}\left[\sqrt{T-t}x_k\varphi_2'(F(\omega, t, \mathbf{x}))\right] = \frac{1}{T-t}\mathrm{Cov}_{\mathbf{x}}\left[\sqrt{T-t}x_k,\ \varphi_2'(F(\omega, t, \mathbf{x}))\right]
\end{aligned}$$

Each equation above gives the rate of change in the equilibrium price of the corresponding asset in terms of the $\mathcal{F}_t$-conditional covariance of the marginal utility of terminal-period wealth (and, thus, consumption), derived from holding an extra unit of the asset, with the future realizations of the kth component of the stochastic process $\beta(\omega, T) - \beta(\omega, t)$. Recall equation (9). Under CARA and either of the terminal-period wealth specifications in (10) or (11), the two rates of change above are the exact same multiples of $p_{A_j}(\omega, t)$ and $p_B(\omega, t)$, respectively.[15] Yet, as the following proposition makes clear, under more general specifications for $F(\cdot)$, the two covariances will not necessarily be the same multiples of the

[15] The specification in (10) nests the case of $F(\omega, t, \mathbf{x})$ being linear in $\beta_k(\omega, t)$. Since the $\mathcal{F}_t$-conditional future realizations $\beta_k(\omega, T) - \beta_k(\omega, t)$ are normally-distributed, this requires unlimited liability making the linearity of the terminal-period wealth in the kth underlying source of uncertainty an unrealistic assumption. Under CARA, the linearity assumption is restrictive also theoretically; it shuts down the asset-riskiness effect.

corresponding equilibrium prices. The extent to which they differ will be purely due to the asset-riskiness effect.

Proposition 4.1.3 *Let the terminal-period utility function $\varphi_2(\cdot)$ exhibit CARA. Suppose also that the matrix of factor loadings σ is such that the following conditions are met.*

(i) The terminal-period dividend $A_j(\omega, T)$ depends only on the mth component of the Brownian process $\beta(\omega, T)$ for some $m \in \{1, ..., d\}$: $\sigma'_j = \sigma_{jm}\mathbf{e}_m$.

(ii) For $k \neq m$, $\beta_k(\omega, T)$ does not affect any component of terminal-period wealth other than the lth ($l \neq j$) terminal-period dividend $A_l(\omega, T)$.

(iii) $A_l(\omega, T)$ depends only on the kth and mth components of $\beta(\omega, T)$: $\sigma'_l = \sigma_{lm}\mathbf{e}_m + \sigma_{lk}\mathbf{e}_k$.

Then, the equilibrium relative price process $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ is monotone in $\beta_k(\omega, t)$. Specifically,

$$\sigma_{lk}\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right) \leq 0 \qquad \text{with equality iff } \sigma_{lk} = 0$$

Example 4.1.2 *Any matrix σ satisfying the conditions of Proposition 4.1.3 is necessarily non-diagonal. The following two matrices are examples where the proposition can be applied for $j = 1$ and $k \in \{2, ..., d\}$. In each case, using also Corollary 4.1, we can sign the entire vector $\nabla_{\beta(\omega,t)}\left(\frac{p_{A_1}(\omega,t)}{p_B(\omega,t)}\right)$.*

$$\begin{pmatrix} \sigma_{11} & 0 \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \qquad \sigma_{11}\tfrac{\partial}{\partial\beta_1(\omega,t)}\left(\tfrac{p_{A_1}(\omega,t)}{p_B(\omega,t)}\right) > 0,\ \sigma_{22}\tfrac{\partial}{\partial\beta_2(\omega,t)}\left(\tfrac{p_{A_1}(\omega,t)}{p_B(\omega,t)}\right) < 0$$

$$\begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & 0 & \sigma_{33} \end{pmatrix} \sigma_{11}\tfrac{\partial}{\partial\beta_1(\omega,t)}\left(\tfrac{p_{A_1}(\omega,t)}{p_B(\omega,t)}\right) > 0,\ \sigma_{kk}\tfrac{\partial}{\partial\beta_k(\omega,t)}\left(\tfrac{p_{A_1}(\omega,t)}{p_B(\omega,t)}\right) < 0\ \ k = 2, 3$$

General Cross-correlations

For general terminal-period wealth and utility specifications, the dynamics of the equilibrium relative price $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ with respect to changes in $\beta_k(\omega, t)$ when $\sigma_{jk} = 0$ are complex.

Proposition 4.1.4 *For $j \in \{1, ..., J\}$, let the terminal-period dividend depend on the realization of only the mth source of uncertainty: $A_j(\omega, T) = e^{\mu_j T + \sigma_{jm}\left(\beta_m(\omega,t) + \sqrt{T-t}x_m\right)}$. For $k \neq m$, we have*

$$\frac{\partial}{\partial \beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)$$

$$= \left(\frac{e^{\mu_j T + \sigma_j \beta(\omega,t)}}{p_B(\omega,t)^2 \sqrt{(T-t)(2\pi)^{2d}}}\right)$$

$$Cov_{y_k}\left[\mathbb{E}_{(\mathbf{x},\mathbf{y}_{-k})}\left[G\left(\mathbf{x}, \mathbf{y} + \sqrt{T-t}\sigma_{jm}\mathbf{e}_j\right) - G\left(\mathbf{x} + \sqrt{T-t}\sigma_{jm}\mathbf{e}_j, \mathbf{y}\right)\right],\ y_k\right]$$

The complexity of the dynamics can be demonstrated in another special yet illustrative case in which $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ depends upon $\beta_k(\omega,t)$ even though the kth component of the stochastic vector $\beta(\omega,T)$ does not affect $A_j(\omega,T)$. For $J = d = 2$, let the matrix of factor loadings σ be diagonal and the terminal-period endowment be some constant $\rho_0 \in \mathbb{R}_+$ so that the terminal-period wealth specification becomes a special case of (12):

$$F(\omega, t, \mathbf{x}) = \rho_0 + \sum_{i=1}^{J} e^{\mu_i T + \sigma_i\left(\beta_i(\omega,t) + \sqrt{T-t}x_i\right)} \tag{13}$$

We return, therefore, to assuming that $\beta_k(\omega,T)$ and the component of the Brownian vector that affects the dividend $A_j(\omega,T)$ operate through different elements of the terminal-period wealth. Yet, the CARA property will be now replaced by DARA. A change in $\beta_k(\omega,t)$ produces wealth effects on the equilibrium prices $p_B(\omega,t)$ and $p_{A_j}(\omega,t)$ whose combination pushes the relative equilibrium price $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ in the same direction as $\sigma_{kk}\mathrm{d}\beta_k(\omega,t)$. This is also the case, moreover, for the asset-riskiness effect of $\mathrm{d}\beta_k(\omega,t)$ on $p_{A_j}(\omega,t)$ (footnote 14).

Proposition 4.1.5 *Let $J = d = 2$ and suppose that*

(i) the terminal-period wealth process is specified as in (13), and

(ii) the terminal-period utility function $\varphi_2(\cdot)$ exhibits DARA.

For $k \neq j$, the equilibrium relative price process $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ is monotone in $\beta_k(\omega,t)$. Specifically,

$$\sigma_{kk}\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right) \geq 0 \quad \textit{with equality iff } \sigma_{kk}=0$$

4.2 Multiple Sources of Uncertainty in the Dividend

Suppose now that the terminal-period dividend of the jth security, $A_j(\omega,T)$, depends on more than one components of the Brownian vector $\beta(\omega,T)$. If it remains invariant to changes in the kth component ($\sigma_{jk}=0$), Proposition 4.1.4 generalizes to the following.

Proposition 4.2.1 *For $j \in \{1,...,J\}$, let $A_j(\omega,T) = e^{\mu_j T + \sigma_j\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}$. If $\sigma_{jk}=0$ for some $k \in \{1,...,d\}$, then*

$$\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right) = \left(\frac{e^{\mu_j T+\sigma_j\beta(\omega,t)}}{p_B(\omega,t)^2\sqrt{(T-t)(2\pi)^{2d}}}\right) Cov_{y_k}\left[\mathbb{E}_{(\mathbf{x},\mathbf{y}_{-k})}\left[G\left(\mathbf{x},\mathbf{y}+\sqrt{T-t}\sigma_j'\right)-G\left(\mathbf{x}+\sqrt{T-t}\sigma_j',\mathbf{y}\right)\right],\ y_k\right]$$

If $\sigma_{jk}\neq 0$, on the other hand, we can establish monotonicity under some restrictions.

Proposition 4.2.2 *Let the terminal-period utility exhibit CRRA: $\varphi_2(c)=\gamma c^{\alpha}$ $\alpha,\gamma<0$.*[16] *Suppose also that*

(i) there is no terminal-period endowment: $\rho(\cdot)=0$, and

(ii) $\sigma_i\sigma_j' = \sigma_j\sigma_j'$ $\forall i \in \{1,...,J\}$.

Then, the equilibrium relative price process $\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}$ is monotone in $\beta_k(\omega,t)$. Specifically,

$$\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right) = \sigma_{jk}\left(\frac{\mathbb{E}_{\mathbf{y}}\left[\varphi_2'(F(\omega,t,\mathbf{y}))\right]}{p_B(\omega,t)}\right)^2 e^{\mu_j T+\sigma_j\beta(\omega,t)+\frac{(2\alpha-1)(T-t)s}{2}}$$

[16] It is trivial to verify (see Section 6.5) that the result applies also to log-utility: $\varphi_2(c)=\ln c$.

Example 4.2.1 *When Proposition 4.2.2 applies, the* j*th row of the Jacobian matrix of equilibrium relative prices is* $\mathbf{J}(\omega,t)_j = \sigma_j \left(\frac{\mathbb{E}_{\mathbf{y}}\left[\varphi_2'(F(\omega,t,\mathbf{y}))\right]}{p_B(\omega,t)}\right)^2 e^{\mu_j T+\sigma_j\beta(\omega,t)+\frac{(2\alpha-1)(T-t)s}{2}}$. *If* $J=d$ *and condition (ii) of Proposition 4.2.2 holds* $\forall j\in\{1,...,J\}$*, the determinant of the Jacobian matrix will be given by* $|\mathbf{J}(\omega,t)| = |\sigma|\prod_{j=1}^{d}\lambda_j$. *Thus, when Proposition 4.2.2 applies to every security in the model and markets are potentially dynamically-complete, they are in fact dynamically-complete as long as the matrix of factor loadings* σ *is non-singular.*

5 Proof of Theorem 2.1

For $j\in\{1,...,J\}$ and $k\in\{1,...,d\}$, we have

$$\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right) = \frac{1}{p_B(\omega,t)}\frac{\partial p_{A_j}(\omega,t)}{\partial\beta_k(\omega,t)} - \frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\frac{\partial p_B(\omega,t)}{\partial\beta_k(\omega,t)} \tag{14}$$

The terms $\frac{\partial p_{A_j}(\omega,t)}{\partial\beta_k(\omega,t)}$ and $\frac{\partial p_B(\omega,t)}{\partial\beta_k(\omega,t)}$ apply the partial-derivative operator $\frac{\partial}{\partial\beta_k(\omega,t)}$ on the expectations $\mathbb{E}_{\mathbf{x}}\left[\varphi_2'(F(\omega,t,\mathbf{x}))\,e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}\right]$ and $\mathbb{E}_{\mathbf{x}}\left[\varphi_2'(F(\omega,t,\mathbf{x}))\right]$, respectively.[17] Lemma A.1 of the Appendix guarantees that the partial-derivative operator commutes with the expectations operator in this case. Before proceeding, I will demonstrate briefly how the lemma can be applied here. For any realization of $(\beta,\mathbf{x})\in\mathbb{R}^d\times\mathbb{R}^d$,

$$\begin{aligned}&\frac{\partial}{\partial\beta_k(\omega,t)}\left(\varphi_2'(F(\omega,t,\mathbf{x}))\,e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}\right)\\ =\;& e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}\left(\sigma_{jk}\varphi_2'(F(\omega,t,\mathbf{x}))+\varphi_2''(F(\omega,t,\mathbf{x}))\frac{\partial F(\omega,t,\mathbf{x})}{\partial\beta_k(\omega,t)}\right)\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial^2}{\partial\beta_k(\omega,t)^2}\left(\varphi_2'(F(\omega,t,\mathbf{x}))\,e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}\right)\\ =\;& e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}\left(\begin{array}{c}\sigma_{jk}^2\varphi_2'(F(\omega,t,\mathbf{x}))\\ +\varphi_2''(F(\omega,t,\mathbf{x}))\left(2\sigma_{jk}\frac{\partial F(\omega,t,\mathbf{x})}{\partial\beta_k(\omega,t)}+\frac{\partial^2 F(\omega,t,\mathbf{x})}{\partial\beta_k(\omega,t)^2}\right)\\ +\varphi_2'''(F(\omega,t,\mathbf{x}))\left(\frac{\partial F(\omega,t,\mathbf{x})}{\partial\beta_k(\omega,t)}\right)^2\end{array}\right)\end{aligned}$$

[17] $p_{A_j}(\omega,t)=\mathbb{E}_{\mathbf{x}}\left[\varphi_2'(F(\omega,t,\mathbf{x}))\,e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}\right]$ and $p_B(\omega,t)=\mathbb{E}_{\mathbf{x}}\left[\varphi_2'(F(\omega,t,\mathbf{x}))\right]$.

Similarly,

$$\begin{aligned}
\frac{\partial}{\partial \beta_k(\omega,t)} \varphi_2'(F(\omega,t,\mathbf{x})) &= \varphi_2''(F(\omega,t,\mathbf{x})) \frac{\partial F(\omega,t,\mathbf{x})}{\partial \beta_k(\omega,t)} \\
\frac{\partial^2}{\partial \beta_k(\omega,t)^2} \varphi_2'(F(\omega,t,\mathbf{x})) &= \varphi_2''(F(\omega,t,\mathbf{x})) \frac{\partial^2 F(\omega,t,\mathbf{x})}{\partial \beta_k(\omega,t)^2} + \varphi_2'''(F(\omega,t,\mathbf{x})) \left(\frac{\partial F(\omega,t,\mathbf{x})}{\partial \beta_k(\omega,t)} \right)^2
\end{aligned}$$

From (1), moreover, we get

$$\begin{aligned}
& \frac{\partial}{\partial \beta_k(\omega,t)} F(\omega,t,\mathbf{x}) \\
&= \frac{\partial}{\partial \beta_k(\omega,t)} \left(\rho\left(\beta(\omega,t) + \sqrt{T-t}\mathbf{x}\right) + \sum_{i=1}^{J} e^{\mu_i T + \sigma_i \left(\beta(\omega,t) + \sqrt{T-t}\mathbf{x}\right)} \right) \\
&= \frac{\partial \rho\left(\beta(\omega,t) + \sqrt{T-t}\mathbf{x}\right)}{\partial \beta_k(\omega,t)} + \sum_{i=1}^{J} \sigma_{ik} e^{\mu_i T + \sigma_i \left(\beta(\omega,t) + \sqrt{T-t}\mathbf{x}\right)} \qquad (15)
\end{aligned}$$

and

$$\frac{\partial^2 F(\omega,t,\mathbf{x})}{\partial \beta_k(\omega,t)^2} = \frac{\partial^2 \rho\left(\beta(\omega,t) + \sqrt{T-t}\mathbf{x}\right)}{\partial \beta_k(\omega,t)^2} + \sum_{i=1}^{J} \sigma_{ik}^2 e^{\mu_i T + \sigma_i \left(\beta(\omega,t) + \sqrt{T-t}\mathbf{x}\right)}$$

Given $\beta_{-k}(\omega,t) \in \mathbb{R}^{d-1}$, take now $F(\omega,t,\mathbf{x})$ as a function of only $\beta_k(\omega,t)$ and $\mathbf{x}$:[18]

$$\begin{aligned}
& F(\beta_k(\omega,t), \mathbf{x}) \\
&= \rho\left((\beta_k(\omega,t), \beta_{-k}(\omega,t)) + \sqrt{T-t}\mathbf{x} \right) + \sum_{i=1}^{J} e^{\mu_i T + \sigma_i \left((\beta_k(\omega,t), \beta_{-k}(\omega,t)) + \sqrt{T-t}\mathbf{x} \right)}
\end{aligned}$$

Define finally $H_1, H_2 : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}_{++}$ by

$$\begin{aligned}
H_1(\beta_k(\omega,t), \mathbf{x}) &= \varphi_2'(F(\beta_k(\omega,t), \mathbf{x})) \\
H_2(\beta_k(\omega,t), \mathbf{x}) &= H_1(\beta_k(\omega,t), \mathbf{x}) \, e^{\mu_j T + \sigma_j \left((\beta_k(\omega,t), \beta_{-k}(\omega,t)) + \sqrt{T-t}\mathbf{x} \right)}
\end{aligned}$$

For all the terminal-period utility functions $\varphi_2(\cdot)$ that are generally of interest in continuous-time finance, it is straightforward to verify that $H_1(\cdot)$ and $H_2(\cdot)$ satisfy the requirements of

[18] For $m \in \{1, ..., d\}$ and $\mathbf{z} \in \mathbb{R}^d$, let $\mathbf{z}_{-m}$ denote the vector $(z_1, ..., z_{-m}, z_{m+1}, ..., z_d)' \in \mathbb{R}^{d-1}$.

Lemma A.1. Therefore, the partial derivative terms in (14) may be written as follows

$$\begin{aligned}\frac{\partial p_{A_j}(\omega,t)}{\partial \beta_k(\omega,t)} &= \mathbb{E}_{\mathbf{x}}\left[\frac{\partial}{\partial \beta_k(\omega,t)}\left(\varphi_2'(F(\omega,t,\mathbf{x}))\, e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}\right)\right] \\ &= \sigma_{jk}\mathbb{E}_{\mathbf{x}}\left[\varphi_2'(F(\omega,t,\mathbf{x}))\, e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}\right] \\ &\quad +\mathbb{E}_{\mathbf{x}}\left[\varphi_2''(F(\omega,t,\mathbf{x}))\, e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}\frac{\partial F(\omega,t,\mathbf{x})}{\partial \beta_k(\omega,t)}\right] \\ \frac{\partial p_B(\omega,t)}{\partial \beta_k(\omega,t)} &= \mathbb{E}_{\mathbf{x}}\left[\varphi_2''(F(\omega,t,\mathbf{x}))\,\frac{\partial F(\omega,t,\mathbf{x})}{\partial \beta_k(\omega,t)}\right]\end{aligned}$$

By Lemma A.2 of the Appendix, moreover,

$$\begin{aligned}\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)} &= e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\frac{(T-t)}{2}\sigma_j'\right)}\frac{\mathbb{E}_{\mathbf{x}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{x}+\sqrt{T-t}\sigma_j'\right)\right)\right]}{\mathbb{E}_{\mathbf{x}}\left[\varphi_2'(F(\omega,t,\mathbf{x}))\right]} \\ \frac{\partial p_{A_j}(\omega,t)}{\partial \beta_k(\omega,t)} &= e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\frac{(T-t)}{2}\sigma_j'\right)} \\ &\quad \left(\begin{array}{c}\sigma_{jk}\mathbb{E}_{\mathbf{x}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{x}+\sqrt{T-t}\sigma_j'\right)\right)\right] \\ +\mathbb{E}_{\mathbf{x}}\left[\varphi_2''\left(F\left(\omega,t,\mathbf{x}+\sqrt{T-t}\sigma_j'\right)\right)\frac{\partial F\left(\omega,t,\mathbf{x}+\sqrt{T-t}\sigma_j'\right)}{\partial \beta_k(\omega,t)}\right]\end{array}\right)\end{aligned}$$

The preceding analysis allows us to rewrite (14) as follows

$$\begin{aligned}& p_B(\omega,t)^2\, e^{-\mu_j T-\sigma_j\beta(\omega,t)}\frac{\partial}{\partial \beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right) \\ =\ & \sigma_{jk}\mathbb{E}_{\mathbf{x}}\left[\varphi_2'(F(\omega,t,\mathbf{x}))\, e^{\sqrt{T-t}\sigma_j\mathbf{x}}\right]\mathbb{E}_{\mathbf{x}}\left[\varphi_2'(F(\omega,t,\mathbf{x}))\right] \\ & + \mathbb{E}_{\mathbf{x}}\left[\varphi_2'(F(\omega,t,\mathbf{x}))\right]\mathbb{E}_{\mathbf{x}}\left[\varphi_2''(F(\omega,t,\mathbf{x}))\, e^{\sqrt{T-t}\sigma_j\mathbf{x}}\frac{\partial F(\omega,t,\mathbf{x})}{\partial \beta_k(\omega,t)}\right] \\ & - \mathbb{E}_{\mathbf{x}}\left[\varphi_2'(F(\omega,t,\mathbf{x}))\, e^{\sqrt{T-t}\sigma_j\mathbf{x}}\right]\mathbb{E}_{\mathbf{x}}\left[\varphi_2''(F(\omega,t,\mathbf{x}))\,\frac{\partial F(\omega,t,\mathbf{x})}{\partial \beta_k(\omega,t)}\right]\end{aligned}$$

Equivalently,

$$\begin{aligned}
& p_B(\omega,t)^2 \, e^{-\mu_j T - \sigma_j \beta(\omega,t)} \frac{\partial}{\partial \beta_k(\omega,t)} \left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)} \right) \\
= \; & \sigma_{jk} \mathbb{E}_{\mathbf{y}} \left[\varphi_2' \left(F(\omega,t,\mathbf{y}) \right) e^{\sqrt{T-t}\sigma_j \mathbf{y}} \right] \mathbb{E}_{\mathbf{x}} \left[\varphi_2' \left(F(\omega,t,\mathbf{x}) \right) \right] \\
& + \mathbb{E}_{\mathbf{x}} \left[\varphi_2' \left(F(\omega,t,\mathbf{x}) \right) \right] \mathbb{E}_{\mathbf{y}} \left[\varphi_2'' \left(F(\omega,t,\mathbf{y}) \right) e^{\sqrt{T-t}\sigma_j \mathbf{y}} \frac{\partial F(\omega,t,\mathbf{y})}{\partial \beta_k(\omega,t)} \right] \qquad (16) \\
& - \mathbb{E}_{\mathbf{y}} \left[\varphi_2' \left(F(\omega,t,\mathbf{y}) \right) e^{\sqrt{T-t}\sigma_j \mathbf{y}} \right] \mathbb{E}_{\mathbf{x}} \left[\varphi_2'' \left(F(\omega,t,\mathbf{x}) \right) \frac{\partial F(\omega,t,\mathbf{x})}{\partial \beta_k(\omega,t)} \right]
\end{aligned}$$

where $\mathbf{y}, \mathbf{x} \sim$ i.i.d. $\mathbf{N}(\mathbf{0}, \mathbf{I}_d)$. From (15), however, for any $\mathbf{x} \in \mathbb{R}^d$ we have

$$\begin{aligned}
\frac{\partial F(\omega,t,\mathbf{x})}{\partial \beta_k(\omega,t)} &= \frac{\partial}{\partial \beta_k(\omega,t)} \rho\left(\beta(\omega,t) + \sqrt{T-t}\mathbf{x} \right) + \sum_{i=1}^{J} \sigma_{ik} e^{\mu_i T + \sigma_i \left(\beta(\omega,t) + \sqrt{T-t}\mathbf{x} \right)} \\
&= \left(\nabla_{\mathbf{z}} \rho(\mathbf{z}) \right)' \mathbf{e}_k \big|_{\mathbf{z} = \beta(\omega,t) + \sqrt{T-t}\mathbf{x}} + \sum_{i=1}^{J} \sigma_{ik} e^{\mu_i T + \sigma_i \left(\beta(\omega,t) + \sqrt{T-t}\mathbf{x} \right)} \\
&= \frac{1}{\sqrt{T-t}} \frac{\partial F(\omega,t,\mathbf{x})}{\partial x_k} \qquad (17)
\end{aligned}$$

By (16) and (17),

$$\begin{aligned}
& p_B(\omega,t)^2 \, e^{-\mu_j T - \sigma_j \beta_j(\omega,t)} \frac{\partial}{\partial \beta_k(\omega,t)} \left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)} \right) \\
= \; & \sigma_{jk} \mathbb{E}_{\mathbf{y}} \left[\varphi_2' \left(F(\omega,t,\mathbf{y}) \right) e^{\sqrt{T-t}\sigma_j \mathbf{y}} \right] \mathbb{E}_{\mathbf{x}} \left[\varphi_2' \left(F(\omega,t,\mathbf{x}) \right) \right] \\
& + \frac{1}{\sqrt{T-t}} \mathbb{E}_{\mathbf{y}} \left[e^{\sqrt{T-t}\sigma_j \mathbf{y}} \frac{\partial}{\partial y_k} \varphi_2' \left(F(\omega,t,\mathbf{y}) \right) \right] \mathbb{E}_{\mathbf{x}} \left[\varphi_2' \left(F(\omega,t,\mathbf{x}) \right) \right] \\
& - \frac{1}{\sqrt{T-t}} \mathbb{E}_{\mathbf{x}} \left[\frac{\partial}{\partial x_k} \varphi_2' \left(F(\omega,t,\mathbf{x}) \right) \right] \mathbb{E}_{\mathbf{y}} \left[\varphi_2' \left(F(\omega,t,\mathbf{y}) \right) e^{\sqrt{T-t}\sigma_j \mathbf{y}} \right]
\end{aligned}$$

It is trivial to check that Lemma A.2 applies on the term $\mathbb{E}_{\mathbf{y}} \left[\varphi_2' \left(F(\omega,t,\mathbf{y}) \right) e^{\sqrt{T-t}\sigma_j \mathbf{y}} \right]$ while Lemma A.3 does so on each of $\mathbb{E}_{\mathbf{y}} \left[e^{\sqrt{T-t}\sigma_j \mathbf{y}} \frac{\partial}{\partial y_k} \varphi_2' \left(F(\omega,t,\mathbf{y}) \right) \right]$ and $\mathbb{E}_{\mathbf{x}} \left[\frac{\partial}{\partial x_k} \varphi_2' \left(F(\omega,t,\mathbf{x}) \right) \right]$.[19]

[19]For the term $\mathbb{E}_{\mathbf{x}} \left[\frac{\partial}{\partial x_k} \varphi_2' \left(F(\omega,t,\mathbf{x}) \right) \right]$, set $\theta = \mathbf{0}$ in Lemma A.3.

Thus,

$$
\begin{aligned}
&\sqrt{T-t}e^{-\mu_j T-\sigma_j\beta(\omega,t)-\frac{(T-t)\sigma_j\sigma_j'}{2}}p_B(\omega,t)^2\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)\\
=\;&\sqrt{T-t}\sigma_{jk}\mathbb{E}_{\mathbf{y}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_j'\right)\right)\right]\mathbb{E}_{\mathbf{x}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)\right]\\
&+\mathbb{E}_{\mathbf{y}}\left[y_k\varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_j'\right)\right)\right]\mathbb{E}_{\mathbf{x}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)\right]\\
&-\mathbb{E}_{\mathbf{x}}\left[x_k\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)\right]\mathbb{E}_{\mathbf{y}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_j'\right)\right)\right]\\
=\;&\mathbb{E}_{\mathbf{y}}\left[\left(y_k+\sqrt{T-t}\sigma_{jk}\right)\varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_j'\right)\right)\right]\mathbb{E}_{\mathbf{x}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)\right]\\
&-\mathbb{E}_{\mathbf{x}}\left[x_k\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)\right]\mathbb{E}_{\mathbf{y}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_j'\right)\right)\right]\\
=\;&\mathbb{E}_{\widetilde{\mathbf{y}}}\left[\widetilde{y}_k\varphi_2'\left(F\left(\omega,t,\widetilde{\mathbf{y}}\right)\right)\right]\mathbb{E}_{\mathbf{x}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)\right]-\mathbb{E}_{\mathbf{x}}\left[x_k\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)\right]\mathbb{E}_{\widetilde{\mathbf{y}}}\left[\varphi_2'\left(F\left(\omega,t,\widetilde{\mathbf{y}}\right)\right)\right]
\end{aligned}
$$

where $\widetilde{\mathbf{y}}\sim\mathbf{N}\left(\sqrt{T-t}\sigma_j',\mathbf{I}_d\right)$ is independent of $\mathbf{x}$. Equivalently,

$$
\begin{aligned}
&\sqrt{(T-t)(2\pi)^{2d}}e^{-\mu_j T-\sigma_j\beta(\omega,t)-\frac{(T-t)\sigma_j\sigma_j'}{2}}p_B(\omega,t)^2\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)\\
=\;&\underset{\mathbb{R}^{2d}}{\int\cdots\int}\varphi_2'\left(F\left(\omega,t,\widetilde{\mathbf{y}}\right)\right)\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)\widetilde{y}_k e^{-\frac{\left(\widetilde{\mathbf{y}}-\sqrt{T-t}\sigma_j'\right)'\left(\widetilde{\mathbf{y}}-\sqrt{T-t}\sigma_j'\right)+\mathbf{x}'\mathbf{x}}{2}}d\mathbf{x}d\widetilde{\mathbf{y}}\\
&-\underset{\mathbb{R}^{2d}}{\int\cdots\int}\varphi_2'\left(F\left(\omega,t,\widetilde{\mathbf{y}}\right)\right)\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)x_k e^{-\frac{\left(\widetilde{\mathbf{y}}-\sqrt{T-t}\sigma_j'\right)'\left(\widetilde{\mathbf{y}}-\sqrt{T-t}\sigma_j'\right)+\mathbf{x}'\mathbf{x}}{2}}d\mathbf{x}d\widetilde{\mathbf{y}}
\end{aligned}
$$

or

$$
\begin{aligned}
&\sqrt{(T-t)(2\pi)^{2d}}e^{-\mu_j T-\sigma_j\beta(\omega,t)}p_B(\omega,t)^2\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)\\
=\;&\underset{\mathbb{R}^{2d}}{\int\cdots\int}\varphi_2'\left(F\left(\omega,t,\mathbf{y}\right)\right)\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)\left(y_k-x_k\right)e^{\sqrt{T-t}\sigma_j\mathbf{y}-\frac{\mathbf{y}'\mathbf{y}+\mathbf{x}'\mathbf{x}}{2}}d\mathbf{x}d\mathbf{y} \qquad (18)
\end{aligned}
$$

Conditional on the filtration $\mathcal{F}_t$, the quantity multiplying $\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)$ on the left-hand side of (18) is a strictly positive constant. That is, $\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)$ is directly proportional to the $2d$-dimensional integral on the right-hand side. Unfortunately, this integral cannot be calculated analytically for general specifications of the functions $\varphi_2(\cdot)$ and $\rho(\cdot)$. Nevertheless, its integrand exhibits a symmetry with respect to the variables of integration so that Lemma

A.4 applies. Specifically, we have two cases.

(i) $\sigma_j' = \mathbf{0}$

In this case, (18) reads

$$\frac{\sqrt{(T-t)(2\pi)^{2d}}p_B(\omega,t)^2}{e^{\mu_j T+\sigma_j\beta(\omega,t)}}\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)=\int\underset{\mathbb{R}^{2d}}{\cdots}\int g(\mathbf{x},\mathbf{y})\,f^0(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}$$

with $g:\mathbb{R}^{2d}\to\mathbb{R}_{++}$ and $f^0:\mathbb{R}^{2d}\to\mathbb{R}$ defined by

$$\begin{aligned} g(\mathbf{x},\mathbf{y}) &= \varphi_2'(F(\omega,t,\mathbf{x}))\,\varphi_2'(F(\omega,t,\mathbf{y}))\,e^{-\frac{\mathbf{y}'\mathbf{y}+\mathbf{x}'\mathbf{x}}{2}} \\ f^0(\mathbf{x},\mathbf{y}) &= \mathbf{e}_k'(\mathbf{y}-\mathbf{x}) \end{aligned}$$

Since $gf^0(\cdot)$ satisfies the requirements for the equality case of Lemma A.4,

$$\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)=0 \qquad \forall k\in\{1,...,d\}$$

(ii) $\sigma_j' \neq \mathbf{0}$

Observe that the quantity multiplying $\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)$ on the left-hand side of (18) is invariant with respect to $k\in\{1,...,d\}$. Summing over k,

$$\begin{aligned} &\frac{\sqrt{(T-t)(2\pi)^{2d}}p_B(\omega,t)^2}{e^{\mu_j T+\sigma_j\beta(\omega,t)}}\sum_{k=1}^{d}\sigma_{jk}\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right) \\ =& \int\underset{\mathbb{R}^{2d}}{\cdots}\int \varphi_2'(F(\omega,t,\mathbf{y}))\,\varphi_2'(F(\omega,t,\mathbf{x}))\,e^{\sqrt{T-t}\sigma_j\mathbf{y}-\frac{\mathbf{y}'\mathbf{y}+\mathbf{x}'\mathbf{x}}{2}}\sum_{k=1}^{d}\sigma_{jk}(y_k-x_k)\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\ =& \int\underset{\mathbb{R}^{2d}}{\cdots}\int g(\mathbf{x},\mathbf{y})\,f(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \end{aligned}$$

where $f : \mathbb{R}^{2d} \to \mathbb{R}$ is given by $f(\mathbf{x},\mathbf{y}) = \sigma_j(\mathbf{y}-\mathbf{x})\, e^{\sqrt{T-t}\sigma_j\mathbf{y}}$. Since $gf(\cdot)$ satisfies the requirements for Lemma A.4, the result follows immediately.[20]

6 Proofs for Section 4

I will now present the proofs for the various results of Section 4. Observe that, by (18), the sign of $\frac{\partial}{\partial\beta_k(\omega,t)}\left(\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)}\right)$ is the same as that of the quantity

$$\begin{aligned}
&\delta_{jk}\\
&= \underset{\mathbb{R}^{2d}}{\int\cdots\int} \varphi_2'(F(\omega,t,\mathbf{y}))\,\varphi_2'(F(\omega,t,\mathbf{x}))\,(y_k-x_k)\, e^{\sqrt{T-t}\sigma_j\mathbf{y}-\frac{\mathbf{y}'\mathbf{y}+\mathbf{x}'\mathbf{x}}{2}}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\\
&= \underset{\mathbb{R}^{2d}}{\int\cdots\int} \varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_j'\right)\right)\varphi_2'(F(\omega,t,\mathbf{x}))\left(y_k+\sqrt{T-t}\sigma_{jk}-x_k\right)\mathrm{d}\Phi(\mathbf{x},\mathbf{y})
\end{aligned}$$

where the last equality applies Lemma A.2.

Recall also that, for the Propositions of Section 4.1, the row-vector of factor loadings for the jth risky security is $\sigma_j = \sigma_{jm}\mathbf{e}_m'$. Thus, for $k \neq m$, we have

$$\delta_{jk} = \underset{\mathbb{R}^{2d}}{\int\cdots\int} \varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_{jm}\mathbf{e}_m\right)\right)\varphi_2'(F(\omega,t,\mathbf{x}))\,(y_k-x_k)\,\mathrm{d}\Phi(\mathbf{x},\mathbf{y})$$

6.1 Propositions 4.1.1 and 4.1.2

Consider a CARA utility function for terminal-period wealth: $\varphi_2(c) = \gamma e^{\alpha c} \quad \gamma,\alpha < 0$.

Proposition 4.1.1 assumes that terminal-period wealth specification in (10). For any

[20] For any $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2d}$, we have
$f(\mathbf{x},\mathbf{y}) + f(\mathbf{y},\mathbf{x}) = \sigma_j(\mathbf{y}-\mathbf{x})\left(e^{\sqrt{T-t}\sigma_j\mathbf{y}} - e^{\sqrt{T-t}\sigma_j\mathbf{x}}\right) = e^{\sqrt{T-t}\sigma_j\mathbf{x}}\sigma_j(\mathbf{y}-\mathbf{x})\left(e^{\sqrt{T-t}\sigma_j(\mathbf{y}-\mathbf{x})}-1\right)$
The term $\sigma_j(\mathbf{y}-\mathbf{x})\left(e^{\sqrt{T-t}\sigma_j(\mathbf{y}-\mathbf{x})}-1\right)$ is strictly positive on all of $\mathbb{R}^{2d}$ except for its zero-measure subset $\left\{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2d} : \sigma_j(\mathbf{y}-\mathbf{x}) = 0\right\}$.

$(\omega,t)\in\Omega\times[0,T]$ and $(\mathbf{x},\mathbf{y})\in\mathbb{R}^d\times\mathbb{R}^d$,

$$\begin{aligned}\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right) &= \alpha\gamma e^{\alpha F_{-k}(\omega,t,\mathbf{x}_{-k})}e^{\alpha F_k(\omega,t,x_k)}\\ \varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_{jm}\mathbf{e}_m\right)\right) &= e^{\alpha\left[F_{-k}\left(\omega,t,\mathbf{y}_{-k}+\sqrt{T-t}\sigma_{jm}\mathbf{e}_{-k,m}\right)-F_{-k}(\omega,t,\mathbf{y}_{-k})\right]}\end{aligned}$$

(abusing notation slightly, $\mathbf{e}_{-k,m}\in\mathbb{R}^{d-1}$ denotes the vector, $\mathbf{z}_{-k}=(z_1,...z_{k-1},z_{k+1},...z_d)'$, which has 1 at its mth entry and zeros everywhere else). We have

$$\frac{\delta_{jk}}{\alpha^2\gamma^2}= \\ \underset{\mathbb{R}^{2(d-1)}}{\int\cdots\int}\left(\int_{\mathbb{R}^2}g^k\left(x_k,y_k\right)f^k\left(x_k,y_k\right)\mathrm{d}\Phi\left(x_k\right)\mathrm{d}\Phi\left(y_k\right)\right)g^{-k}\left(\mathbf{x}_{-k},\mathbf{y}_{-k}\right)\mathrm{d}\Phi\left(\mathbf{x}_{-m}\right)\mathrm{d}\Phi\left(\mathbf{y}_{-k}\right)$$

where $g^{-k}:\mathbb{R}^{d-1}\times\mathbb{R}^{d-1}\to\mathbb{R}_{++}$, $g^k:\mathbb{R}^2\to\mathbb{R}_{++}$, and $f^k:\mathbb{R}^2\to\mathbb{R}$ are defined by

$$\begin{aligned}g^{-k}\left(\mathbf{x}_{-k},\mathbf{y}_{-k}\right) &= e^{\alpha\left[F_{-k}\left(\omega,t,\mathbf{y}_{-k}+\sqrt{T-t}\sigma_{jm}\mathbf{e}_{-k,m}\right)+F_{-k}(\omega,t,\mathbf{x}_{-k})\right]}\\ g^k\left(x_k,y_k\right) &= e^{\alpha[F_k(\omega,t,x_k)+F_k(\omega,t,y_k)]}\\ f^k\in\mathbb{R}^{d-1} &= y_k-x_k\end{aligned}$$

It is straightforward to check that the equality case of Lemma A.4 in the Appendix applies for the two-dimensional integral in brackets.

Proposition 4.1.2 takes the terminal-period wealth specification in (11). For any $(\omega,t)\in\Omega\times[0,T]$, and $(\mathbf{x},\mathbf{y})\in\mathbb{R}^d\times\mathbb{R}^d$,

$$\begin{aligned}\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right) &= \alpha\gamma e^{\alpha F_{-m}(\omega,t,\mathbf{x}_{-m})}e^{\alpha F_m(\omega,t,\mathbf{x}_m)}\\ \varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_{jm}\mathbf{e}_m\right)\right) &= e^{\alpha\left[F_m\left(\omega,t,y_m+\sqrt{T-t}\sigma_{jm}\right)-F_m(\omega,t,y_m)\right]}\end{aligned}$$

We now get

$$\frac{\delta_{jk}}{\alpha^2\gamma^2} =$$

$$\int_{\mathbb{R}^2} g^m(x_m, y_m) \left(\int \cdots \int_{\mathbb{R}^{2(d-1)}} g^{-m} f^{-m}(\mathbf{x}_{-m}, \mathbf{y}_{-m})\, \mathrm{d}\Phi(\mathbf{x}_{-m})\, \mathrm{d}\Phi(\mathbf{y}_{-m}) \right) \mathrm{d}\Phi(x_m)\, \mathrm{d}\Phi(y_m)$$

where the functions $g^{-m} : \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \to \mathbb{R}_{++}$, $f^{-m} : \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \to \mathbb{R}$, and $g^m : \mathbb{R}^2 \to \mathbb{R}_{++}$ are defined by

$$\begin{aligned} g^{-m}(\mathbf{x}_{-m}, \mathbf{y}_{-m}) &= e^{\alpha[F_{-m}(\omega,t,\mathbf{x}_{-m}) + F_{-m}(\omega,t,\mathbf{y}_{-m})]} \\ g^m(x_m, y_m) &= e^{\alpha\left[F_m(\omega,t,x_m) + F_m\left(\omega,t,y_m + \sqrt{T-t}\sigma_{jm}\right)\right]} \\ f^{-m}(\mathbf{x}_{-m}, \mathbf{y}_{-m}) &= y_k - x_k \end{aligned}$$

while $g^{-m}f^{-m}(\cdot)$ denotes the product function $g^{-m}(\cdot)\, f^{-m}(\cdot)$. The equality case of Lemma A.4 applies for the $2(d-1)$-dimensional integral in the brackets.

6.2 Propositions 4.1.4 and 4.2.1

For Proposition 4.1.4, by renaming the variables of integration, δ_{jk} can be written as follows

$$\begin{aligned} \delta_{jk} &= \int \cdots \int_{\mathbb{R}^{2d}} \varphi_2'\left(F\left(\omega, t, \mathbf{y} + \sqrt{T-t}\sigma_{jm}\mathbf{e}_m\right)\right) \varphi_2'(F(\omega,t,\mathbf{x}))\, y_k \mathrm{d}\Phi(\mathbf{x},\mathbf{y}) \\ &\quad - \int \cdots \int_{\mathbb{R}^{2d}} \varphi_2'\left(F\left(\omega, t, \mathbf{x} + \sqrt{T-t}\sigma_{jm}\mathbf{e}_m\right)\right) \varphi_2'(F(\omega,t,\mathbf{y}))\, y_k \mathrm{d}\Phi(\mathbf{x},\mathbf{y}) \\ &= \int \cdots \int_{\mathbb{R}^{2d}} \left(G\left(\mathbf{x}, \mathbf{y} + \sqrt{T-t}\sigma_{jm}\mathbf{e}_m\right) - G\left(\mathbf{x} + \sqrt{T-t}\sigma_{jm}\mathbf{e}_m, \mathbf{y}\right)\right) y_k \mathrm{d}\Phi(\mathbf{x},\mathbf{y}) \\ &= \mathbb{E}_{(\mathbf{x},\mathbf{y})}\left[\left(G\left(\mathbf{x}, \mathbf{y} + \sqrt{T-t}\sigma_{jm}\mathbf{e}_m\right) - G\left(\mathbf{x} + \sqrt{T-t}\sigma_{jm}\mathbf{e}_m, \mathbf{y}\right)\right) y_k\right] \\ &= \mathbb{E}_{y_k}\left[\mathbb{E}_{(\mathbf{x},\mathbf{y}_{-k})}\left[G\left(\mathbf{x}, \mathbf{y} + \sqrt{T-t}\sigma_{jm}\mathbf{e}_m\right) - G\left(\mathbf{x} + \sqrt{T-t}\sigma_{jm}\mathbf{e}_m, \mathbf{y}\right)\right] y_k\right] \end{aligned}$$

Since $\mathbb{E}_{y_k}[y_k] = 0$, the claim follows immediately.

Regarding Proposition 4.2.1, as long as $\sigma_{jk} = 0$ for $k \neq m$, an identical argument can be made if $\sigma_{jm}\mathbf{e}_m$ above is replaced by σ'_j.

6.3 Proposition 4.1.3

Notice that

$$\mathbb{E}_{(\mathbf{x},\mathbf{y})}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{y}\right)\right)\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)\left(y_k-x_k\right)\right]=0$$

which, by renaming the variables of integration in $\mathbf{y}_{-k}\in\mathbb{R}^{d-1}$, can be re-written as follows

$$\mathbb{E}_{(\mathbf{x},(\mathbf{z}_{-k},y_k))}\left[\varphi_2'\left(F\left(\omega,t,(\mathbf{z}_{-k},y_k)\right)\right)\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)\left(y_k-x_k\right)\right]=0$$

Therefore, in general, we can write

$$\begin{aligned}
&\delta_{jk}\\
=\ &\mathbb{E}_{(\mathbf{x},\mathbf{y})}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_j'\right)\right)\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)\left(y_k+\sqrt{T-t}\sigma_{jk}-x_k\right)\right]\\
&-\mathbb{E}_{(\mathbf{x},(\mathbf{z}_{-k},y_k))}\left[\varphi_2'\left(F\left(\omega,t,(\mathbf{z}_{-k},y_k)\right)\right)\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)\left(y_k-x_k\right)\right]\\
=\ &\mathbb{E}_{y_k}\left[\mathbb{E}_{\mathbf{x}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)\left(y_k+\sqrt{T-t}\sigma_{jk}-x_k\right)\right]\left(\begin{array}{c}\mathbb{E}_{\mathbf{y}_{-k}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_j'\right)\right)\right]\\-\mathbb{E}_{\mathbf{z}_{-k}}\left[\varphi_2'\left(F\left(\omega,t,(\mathbf{z}_{-k},y_k)\right)\right)\right]\end{array}\right)\right]\\
=\ &\mathbb{E}_{y_k}\left[\begin{array}{c}\left(\frac{\mathbb{E}_{\mathbf{y}_{-k}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_j'\right)\right)\right]}{\mathbb{E}_{\mathbf{z}_{-k}}\left[\varphi_2'\left(F\left(\omega,t,(\mathbf{z}_{-k},y_k)\right)\right)\right]}-1\right)\\ \mathbb{E}_{\mathbf{x}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)\left(y_k+\sqrt{T-t}\sigma_{jk}-x_k\right)\right]\mathbb{E}_{\mathbf{z}_{-k}}\left[\varphi_2'\left(F\left(\omega,t,(\mathbf{z}_{-k},y_k)\right)\right)\right]\end{array}\right]
\end{aligned}\tag{19}$$

Let $\sigma_{jk}=0$ and consider $\mathbb{E}_{\mathbf{x}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)\left(y_k-x_k\right)\right]\mathbb{E}_{\mathbf{z}_{-k}}\left[\varphi_2'\left(F\left(\omega,t,(\mathbf{z}_{-k},y_k)\right)\right)\right]$ and $\frac{\mathbb{E}_{\mathbf{y}_{-k}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_{jj}\mathbf{e}_j\right)\right)\right]}{\mathbb{E}_{\mathbf{z}_{-k}}\left[\varphi_2'\left(F\left(\omega,t,(\mathbf{z}_{-k},y_k)\right)\right)\right]}$ as functions of $y_k\in\mathbb{R}$, denoting them by $\psi_1:\mathbb{R}\to\mathbb{R}_{++}$ and $\psi_2:\mathbb{R}\to\mathbb{R}$, respectively. Under the conditions of Proposition 4.1.3, the function $\psi_1\left(y_k\right)=\frac{\mathbb{E}_{\mathbf{y}_{-k}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_j'\right)\right)\right]}{\mathbb{E}_{\mathbf{z}_{-k}}\left[\varphi_2'\left(F\left(\omega,t,(\mathbf{z}_{-k},y_k)\right)\right)\right]}$ is strictly decreasing (strictly increasing) on $\mathbb{R}$ if $\sigma_{ik}>0$ ($\sigma_{ik}<0$).

Claim 6.3.1 *Suppose that the following conditions are met.*

(i) The terminal-period utility function exhibits CARA: $\varphi_2\left(c\right)=\gamma e^{\alpha c}\ \gamma,\alpha<0$.

(ii) $A_j(\omega, T)$ *depends only on the mth component of* $\beta(\omega, t)$*, for some* $m \in \{1, ..., d\}$*:* $\sigma_j' = \sigma_{jm}\mathbf{e}_m$.

(iii) The kth ($k \neq m$) *component of the Brownian process,* $\beta_k(\omega, t)$*, does not affect any component of terminal-period wealth other than the lth dividend,* $A_l(\omega, T)$*, for some* $l \in \{1, ..., J\} \setminus \{j\}$.

(iv) $A_l(\omega, T)$ *depends only on the kth and mth components of* $\beta(\omega, t)$*:* $\sigma_l' = \sigma_{lm}\mathbf{e}_m + \sigma_{lk}\mathbf{e}_k$.

Then, $\psi_1(\cdot)$ *is strictly monotone on* $\mathbb{R}$*:* $\sigma_{lk}\psi_1'(y_k) < 0$, $\forall y_k \in \mathbb{R}$.

Proof. At any $y_k \in \mathbb{R}$, $\psi_1'(\cdot)$ has the same sign as the following quantity

$$
\begin{aligned}
& I(y_k) \\
= \;& \mathbb{E}_{\mathbf{y}_{-k}}\left[\varphi_2''\left(F\left(\omega, t, \mathbf{y} + \sqrt{T-t}\sigma_j'\right)\right)\frac{\partial F\left(\omega, t, \mathbf{y} + \sqrt{T-t}\sigma_j'\right)}{\partial y_k}\right]\mathbb{E}_{\mathbf{z}_{-k}}\left[\varphi_2'\left(F\left(\omega, t, (\mathbf{z}_{-k}, y_k)\right)\right)\right] \\
& -\mathbb{E}_{\mathbf{y}_{-k}}\left[\varphi_2'\left(F\left(\omega, t, \mathbf{y} + \sqrt{T-t}\sigma_j'\right)\right)\right]\mathbb{E}_{\mathbf{z}_{-k}}\left[\varphi_2''\left(F\left(\omega, t, (\mathbf{z}_{-k}, y_k)\right)\right)\frac{\partial F\left(\omega, t, (\mathbf{z}_{-k}, y_k)\right)}{\partial y_k}\right] \\
= \;& \mathbb{E}_{(\mathbf{z}_{-k}, \mathbf{y}_{-k})}\left[\begin{array}{c} \varphi_2''\left(F\left(\omega, t, \mathbf{y} + \sqrt{T-t}\sigma_j'\right)\right)\varphi_2'\left(F\left(\omega, t, (\mathbf{z}_{-k}, y_{-k})\right)\right)\frac{\partial F\left(\omega, t, \mathbf{y} + \sqrt{T-t}\sigma_j'\right)}{\partial y_k} \\ -\varphi_2'\left(F\left(\omega, t, \mathbf{y} + \sqrt{T-t}\sigma_j'\right)\right)\varphi_2''\left(F\left(\omega, t, (\mathbf{z}_{-k}, y_k)\right)\right)\frac{\partial F(\omega, t, (\mathbf{z}_{-k}, y_k))}{\partial y_k} \end{array}\right] \\
= \;& r_A e^{-\frac{(T-t)\sigma_j\sigma_j'}{2}}\mathbb{E}_{(\mathbf{z}_{-k}, \mathbf{y}_{-k})}\left[\begin{array}{c} \varphi_2'\left(F\left(\omega, t, \mathbf{y}\right)\right)\varphi_2'\left(F\left(\omega, t, (\mathbf{z}_{-k}, y_k)\right)\right) \\ \left(\frac{\partial F(\omega, t, (\mathbf{z}_{-k}, y_k))}{\partial y_k} - \frac{\partial F(\omega, t, \mathbf{y})}{\partial y_k}\right)e^{\sqrt{T-t}\sigma_j\mathbf{y}} \end{array}\right]
\end{aligned}
$$

where the last equality above applies Lemma A.2 ($\sigma_{jk} = 0$) and $r_A : \mathbb{R}_{++} \to \mathbb{R}_{++}$ denotes the coefficient of absolute risk aversion, $r_A(\cdot) = -\frac{\varphi_2''(\cdot)}{\varphi_2'(\cdot)}$. Recall now the definition of terminal-period wealth in (1). For any $\mathbf{x} \in \mathbb{R}^d$, we have

$$
\frac{\partial F(\omega, t, \mathbf{x})}{\sqrt{T-t}\partial x_k} = \sum_{i=1}^{J}\sigma_{ik}e^{\mu_i T + \sigma_i\left(\beta(\omega, t) + \sqrt{T-t}\mathbf{x}\right)} = \sigma_{lk}e^{\mu_l T + \sigma_l\beta(\omega, t) + \sqrt{T-t}(\sigma_{lk}x_{lk} + \sigma_{lm}x_{lm})}
$$

Therefore,

$$\begin{aligned}
\frac{e^{\frac{(T-t)\sigma_j\sigma_j'}{2}} I(y_k)}{\sqrt{T-t}r_A} &= \sigma_{lk} e^{\mu_l T+\sigma_l\beta(\omega,t)+\sqrt{T-t}\sigma_{lk}y_k} \\
&\quad \mathbb{E}_{(\mathbf{z}_{-k},\mathbf{y}_{-k})}\left[G(\mathbf{y},(\mathbf{z}_{-k},y_k))\left(e^{\sqrt{T-t}\sigma_{jm}z_m}-e^{\sqrt{T-t}\sigma_{jm}y_m}\right)e^{\sqrt{T-t}\sigma_{jm}y_m}\right] \\
&= \sigma_{lk} e^{\mu_l T+\sigma_l\beta(\omega,t)+\sqrt{T-t}\sigma_{lk}y_k} \underset{\mathbb{R}^{2d-1}}{\int\cdots\int} G(\mathbf{y},(\mathbf{z}_{-k},y_k)) f(y_m,z_m)\,\mathrm{d}\Phi(\mathbf{y}_{-k},\mathbf{z}_{-k})
\end{aligned}$$

where $f:\mathbb{R}^2\to\mathbb{R}$ is defined by

$$f(y_m,z_m)=e^{\sqrt{T-t}\sigma_{jm}y_m}\left(e^{\sqrt{T-t}\sigma_{jm}z_m}-e^{\sqrt{T-t}\sigma_{jm}y_m}\right)$$

while $G:\mathbb{R}^{2d}\to\mathbb{R}_{++}$ is as in the statement of Theorem 2.1. Since $f(y_m,z_m)+f(z_m,y_m)=-\left(e^{\sqrt{T-t}\sigma_{jm}y_m}-e^{\sqrt{T-t}\sigma_{jm}z_m}\right)^2$, by Lemma A.4, the last integral above is always negative. ■

We can conclude now that $\sigma_{lk}\delta_{jk}<0$. To see this, let $\sigma_{lk}>0$ (a trivially similar argument applies to the case $\sigma_{lk}<0$). By Lemma A.5 of the Appendix, there exists $\lambda_k\in\mathbb{R}$ such that $(y_k-\lambda_k)\psi_2(y_k)>0$, $\forall y_k\in\mathbb{R}$. Hence,

$$\begin{aligned}
\delta_{jk} &= \mathbb{E}_{y_k}\left[(\psi_1(y_k)-1)\psi_2(y_k)\right] \\
&= \int_{\mathbb{R}}[\psi_1(y_k)-1]\psi_2(y_k)\,\mathrm{d}\Phi(y_k) \\
&< \int_{y_k\in(\lambda_k,+\infty)}[\psi_1(\lambda_k)-1]\psi_2(y_k)\,\mathrm{d}\Phi(y_k)+\int_{y_k\in(-\infty,\lambda_k)}[\psi_1(\lambda_k)-1]\psi_2(y_k)\,\mathrm{d}\Phi(y_k) \\
&= [\psi_1(\lambda_k)-1]\int_{y_k\in\mathbb{R}}\psi_2(y_k)\,\mathrm{d}\Phi(y_k)
\end{aligned}$$

To complete the argument, note that $\mathbb{E}_{y_k}[\psi_2(y_k)]=0$.

6.4 Proposition 4.1.5

Under the terminal-period wealth specification in (12), the matrix of factor loadings σ is diagonal. Let $\{j,k\}$ denote the two securities (equivalently, the two sources of uncertainty).

Equation (19) reads

$$\delta_{jk} = \mathbb{E}_{y_k}\left[\frac{\left(\frac{\mathbb{E}_{y_j}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_{jj}\mathbf{e}_j\right)\right)\right]}{\mathbb{E}_{z_j}\left[\varphi_2'\left(F\left(\omega,t,(z_j,y_k)\right)\right)\right]}-1\right)}{\mathbb{E}_{\mathbf{x}}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{x}\right)\right)\left(y_k-x_k\right)\right]\mathbb{E}_{z_j}\left[\varphi_2'\left(F\left(\omega,t,(z_j,y_k)\right)\right)\right]}\right]$$

The proof proceeds in exactly the same way as in Section 6.3. The following claim establishes that the function $\psi_1(\cdot)$ is now strictly increasing (decreasing) on $\mathbb{R}$ if $\sigma_{kk} > 0$ ($\sigma_{kk} < 0$).

Claim 6.4.1 *Suppose that the terminal-period wealth is given by (13) with $J = d = 2$. Let also $\psi_1 : \mathbb{R} \to \mathbb{R}_{++}$ be defined by*

$$\psi_1(y_k) = \frac{\mathbb{E}_{y_j}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_{jj}\mathbf{e}_j\right)\right)\right]}{\mathbb{E}_{z_j}\left[\varphi_2'\left(F\left(\omega,t,(z_j,y_k)\right)\right)\right]}$$

Then, $\psi_1(\cdot)$ is strictly monotone on $\mathbb{R}$: $\sigma_{kk}\psi_1'(y_k) > 0$, $\forall y_k \in \mathbb{R}$.

Proof. Given (13), at any $y_k \in \mathbb{R}$, $\psi_1'(\cdot)$ has the same sign as the following quantity

$$\begin{aligned}
&\left(\begin{array}{c}\mathbb{E}_{y_j}\left[\varphi_2''\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_{jj}\mathbf{e}_j\right)\right)\right]\mathbb{E}_{z_j}\left[\varphi_2'\left(F\left(\omega,t,(z_j,y_k)\right)\right)\right]\\ -\mathbb{E}_{y_j}\left[\varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_{jj}\mathbf{e}_j\right)\right)\right]\mathbb{E}_{z_j}\left[\varphi_2''\left(F\left(\omega,t,(z_j,y_k)\right)\right)\right]\end{array}\right)\frac{\partial F_k(\omega,t,y_k)}{\partial y_k}\\
=\;& \mathbb{E}_{(z_j,y_j)}\left[\begin{array}{c}\varphi_2''\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_{jj}\mathbf{e}_j\right)\right)\varphi_2'\left(F\left(\omega,t,(z_j,y_k)\right)\right)\\ -\varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_{jj}\mathbf{e}_j\right)\right)\varphi_2''\left(F\left(\omega,t,(z_j,y_k)\right)\right)\end{array}\right]\frac{\partial F_k(\omega,t,y_k)}{\partial y_k}\\
=\;& \mathbb{E}_{(z_j,y_j)}\left[\begin{array}{c}\varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_{jj}\mathbf{e}_j\right)\right)\varphi_2'\left(F\left(\omega,t,(z_j,y_k)\right)\right)\\ \left[r_A\left(F\left(\omega,t,(z_j,y_k)\right)\right)-r_A\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_{jj}\mathbf{e}_j\right)\right)\right]\end{array}\right]\frac{\partial F_k(\omega,t,y_k)}{\partial y_k}
\end{aligned}$$

Consider the integral

$$\begin{aligned} I\left(y_k\right) &= \mathbb{E}_{\left(z_j,y_j\right)}\left[\begin{array}{c} \varphi_2'\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_{jj}\mathbf{e}_j\right)\right)\varphi_2'\left(F\left(\omega,t,\left(z_j,y_k\right)\right)\right) \\ \left[r_A\left(F\left(\omega,t,\left(z_j,y_k\right)\right)\right)-r_A\left(F\left(\omega,t,\mathbf{y}+\sqrt{T-t}\sigma_{jj}\mathbf{e}_j\right)\right)\right] \end{array}\right] \\ &= e^{-\frac{(T-t)\sigma_{jj}^2}{2}}\mathbb{E}_{\left(z_j,y_j\right)}\left[\begin{array}{c} \varphi_2'\left(F\left(\omega,t,\mathbf{y}\right)\right)\varphi_2'\left(F\left(\omega,t,\left(z_j,y_k\right)\right)\right) \\ \left[r_A\left(F\left(\omega,t,\left(z_j,y_k\right)\right)\right)-r_A\left(F\left(\omega,t,\mathbf{y}\right)\right)\right]e^{\sqrt{T-t}\sigma_{jj}y_j} \end{array}\right] \\ &= e^{-\frac{(T-t)\sigma_{jj}^2}{2}}\underset{\mathbb{R}^{2(d-1)}}{\int\cdots\int} G\left(\omega,t,\mathbf{y},\left(z_j,y_k\right)\right)\widetilde{f}\left(\omega,t,\mathbf{y},\left(z_j,y_k\right)\right)\mathrm{d}y_j\mathrm{d}z_j \end{aligned}$$

where the second equality above applies Lemma A.2. and $\widetilde{f}:\mathbb{R}^3\to\mathbb{R}$ is given by

$$\widetilde{f}\left(\omega,t,\mathbf{y},\left(z_j,y_k\right)\right) = \left[r_A\left(\omega,t,\left(z_j,y_k\right)\right)-r_A\left(\omega,t,\mathbf{y}\right)\right]e^{\sqrt{T-t}\sigma_{jj}y_j}$$

By the DARA property,[21]

$$\begin{aligned} & \widetilde{f}\left(\omega,t,\mathbf{y},\left(z_j,y_k\right)\right)+\widetilde{f}\left(\omega,t,\left(z_j,y_k\right),\mathbf{y}\right) \\ = & \left[r_A\left(F\left(\omega,t,\left(z_j,y_k\right)\right)\right)-r_A\left(F\left(\omega,t,\left(y_j,y_k\right)\right)\right)\right]\left[e^{\sqrt{T-t}\sigma_{jj}y_j}-e^{\sqrt{T-t}\sigma_{jj}z_j}\right] \\ > & \ 0 \end{aligned}$$

and, thus, Lemma A.4 can be applied on the last integral above. Hence, $I\left(y_k\right)>0$ for any $y_k\in\mathbb{R}$ and $\psi_1'\left(y_k\right)$ has the same sign as the quantity $I\left(y_k\right)\frac{\partial F_k(\omega,t,y_k)}{\partial y_k}$. But, under (12), $\frac{\partial F_k(\omega,t,y_k)}{\partial y_k}=\sigma_{kk}e^{\mu_k T+\sigma_{kk}\left(\beta_k(\omega,t)+\sqrt{T-t}y_k\right)}$. ■

To complete the proof, by showing that $\sigma_{kk}\delta_{jk}>0$, the remaining argument is trivially similar to the last step in the proof of Proposition 4.1.3.

[21] $\frac{\partial}{\partial y_j}r_A\left(F\left(\omega,t,\left(y_j,y_k\right)\right)\right)=r_A'\left(F\left(\omega,t,\left(y_j,y_k\right)\right)\right)\frac{\partial F_j(\omega,t,y_j)}{\partial y_j}=\sigma_{jj}r_A'\left(F\left(\omega,t,\left(y_j,y_k\right)\right)\right)e^{\sqrt{T-t}\sigma_{jj}y_j}$ has the opposite sign of σ_{jj} for any $\left(y_j,y_k\right)\in\mathbb{R}^2$.

6.5 Proposition 4.2.2

Let $\rho(\cdot) = 0$ and $\sigma_i \sigma_j' = s > 0 \ \ \forall i \in \{1, ..., J\}$. By (1), we get

$$\begin{aligned} & F\left(\omega, t, \mathbf{x} + \sqrt{T-t}\sigma_j'\right) \\ = & \sum_{i=1}^{J} e^{(T-t)\sigma_i\sigma_j'} e^{\mu_i T + \sigma_i\left(\beta(\omega,t) + \sqrt{T-t}\mathbf{x}\right)} = e^{(T-t)s} \sum_{i=1}^{J} e^{\mu_i T + \sigma_i\left(\beta(\omega,t) + \sqrt{T-t}\mathbf{x}\right)} \end{aligned}$$

and $\varphi_2'\left(F\left(\omega, t, \mathbf{x} + \sqrt{T-t}\sigma_j'\right)\right) = e^{(\alpha-1)(T-t)s} \varphi_2'\left(F\left(\omega, t, \mathbf{x}\right)\right)$. By Section 5, moreover,

$$\begin{aligned} & \sqrt{T-t} e^{-\mu_j T - \sigma_j \beta(\omega,t) - \frac{(T-t)\sigma_j\sigma_j'}{2}} p_B(\omega, t)^2 \frac{\partial}{\partial \beta_k(\omega, t)} \left(\frac{p_{A_j}(\omega, t)}{p_B(\omega, t)}\right) \\ = & \mathbb{E}_{\mathbf{y}}\left[\left(y_k + \sqrt{T-t}\sigma_{jk}\right) \varphi_2'\left(F\left(\omega, t, \mathbf{y} + \sqrt{T-t}\sigma_j'\right)\right)\right] \mathbb{E}_{\mathbf{x}}\left[\varphi_2'\left(F\left(\omega, t, \mathbf{x}\right)\right)\right] \\ & - \mathbb{E}_{\mathbf{x}}\left[x_k \varphi_2'\left(F\left(\omega, t, \mathbf{x}\right)\right)\right] \mathbb{E}_{\mathbf{y}}\left[\varphi_2'\left(F\left(\omega, t, \mathbf{y} + \sqrt{T-t}\sigma_j'\right)\right)\right] \end{aligned}$$

In this case, therefore,

$$\begin{aligned} & \frac{\sqrt{T-t} p_B(\omega, t)^2}{e^{\mu_j T + \sigma_j \beta(\omega,t) + \frac{(2\alpha-1)(T-t)s}{2}}} \frac{\partial}{\partial \beta_k(\omega, t)} \left(\frac{p_{A_j}(\omega, t)}{p_B(\omega, t)}\right) \\ = & \sqrt{T-t}\sigma_{jk} \mathbb{E}_{\mathbf{y}}\left[\varphi_2'\left(F\left(\omega, t, \mathbf{y}\right)\right)\right]^2 + \mathbb{E}_{(\mathbf{x},\mathbf{y})}\left[\varphi_2'\left(F\left(\omega, t, \mathbf{x}\right)\right) \varphi_2'\left(F\left(\omega, t, \mathbf{y}\right)\right)\left(y_k - x_k\right)\right] \\ = & \sqrt{T-t}\sigma_{jk} \mathbb{E}_{\mathbf{y}}\left[\varphi_2'\left(F\left(\omega, t, \mathbf{y}\right)\right)\right]^2 \end{aligned}$$

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A Lemmata

Lemma A.1 *Let* $H : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ *be* 2^{nd}*-order differentiable. Suppose that, for* $\epsilon > 0$*,*

$$\sup_{\widetilde{\beta}\in[\beta-\epsilon,\beta+\epsilon]} \sup_{\mathbf{x}\in\mathbb{R}^d} \left| \frac{\partial^2}{\partial\beta^2} H\left(\widetilde{\beta}, \mathbf{x}\right) \right| \prod_{k=1}^{d} \max\left\{1, x_k^2\right\} \exp\left(-\frac{\mathbf{x}'\mathbf{x}}{2}\right) = K(\beta) < \infty$$

and that $G : \mathbb{R} \longrightarrow \mathbb{R}$ *given by*

$$G\left(\widetilde{\beta}\right) = \underset{\mathbb{R}^d}{\int \cdots \int} H\left(\widetilde{\beta}, \mathbf{x}\right) d\Phi(\mathbf{x}) = \mathbb{E}_{\mathbf{x}}\left[H\left(\widetilde{\beta}, \mathbf{x}\right)\right]$$

is defined for all $\widetilde{\beta} \in [\beta - \epsilon, \beta + \epsilon]$*. Then* G *is differentiable at* β *and*

$$G'(\beta) = \underset{\mathbb{R}^d}{\int \cdots \int} \frac{\partial H(\beta, \mathbf{x})}{\partial\beta} d\Phi(\mathbf{x}) = \mathbb{E}_{\mathbf{x}}\left[\frac{\partial H(\beta, \mathbf{x})}{\partial\beta}\right]$$

Proof. For $h \in \mathbb{R} \setminus \{0\} : |h| < \epsilon$, we have

$$
\begin{aligned}
& \left| \frac{G(\beta+h) - G(\beta)}{h} - \int_{\mathbb{R}^d} \cdots \int \frac{\partial H(\beta, \mathbf{x})}{\partial \beta} \mathrm{d}\Phi(\mathbf{x}) \right| \\
= & \left| \int_{\mathbb{R}^d} \cdots \int \left(\frac{H(\beta+h, \mathbf{x}) - H(\beta, \mathbf{x})}{h} - \frac{\partial H(\beta, \mathbf{x})}{\partial \beta} \right) \mathrm{d}\Phi(\mathbf{x}) \right| \\
\leq & \int_{\mathbb{R}^d} \cdots \int \left| \frac{H(\beta+h, \mathbf{x}) - H(\beta, \mathbf{x})}{h} - \frac{\partial H(\beta, \mathbf{x})}{\partial \beta} \right| \mathrm{d}\Phi(\mathbf{x}) \\
= & \int_{\mathbb{R}^d} \cdots \int \left| \frac{\partial H(\beta + \gamma(\mathbf{x}) h, \mathbf{x})}{\partial \beta H(\beta, \mathbf{x})} - \frac{\partial}{\partial \beta} \right| \mathrm{d}\Phi(\mathbf{x}) \quad \text{for some } \gamma(\mathbf{x}) \in (0,1) \\
= & |h| \int_{\mathbb{R}^d} \cdots \int |\gamma(\mathbf{x})| \left| \frac{\partial^2 H(\beta + \delta(\mathbf{x}) \gamma(\mathbf{x}) h, \mathbf{x})}{\partial \beta^2} \right| \mathrm{d}\Phi(\mathbf{x}) \quad \text{for some } \delta(\mathbf{x}) \in (0,1) \\
< & |h| \int_{\mathbb{R}^d} \cdots \int \left| \frac{\partial^2 H(\beta + \delta(\mathbf{x}) \gamma(\mathbf{x}) h, \mathbf{x})}{\partial \beta^2} \right| \mathrm{d}\Phi(\mathbf{x}) \\
\leq & \frac{|h| K(\beta)}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} \cdots \int \frac{\mathrm{d}\mathbf{x}}{\prod_{k=1}^{d} \max\{1, x_k^2\}} = \left(\frac{4}{\sqrt{2\pi}} \right)^d |h| K(\beta)
\end{aligned}
$$

where the second and third equalities above follow from the Mean-value Theorem. Taking $|h| \to 0$ gives the required result.[22] ■

For completeness, I include the following (well-known in statistics) result without proof.

Lemma A.2 *Let $\mathbf{z} \in \mathbb{R}^d$ be a random vector distributed $N(\mathbf{0}, \mathbf{I}_d)$ and $\theta \in \mathbb{R}^d$. Suppose $g : \mathbb{R}^d \to \mathbb{R}$ is such that $\mathbb{E}_{\mathbf{z}}\left[e^{\theta' \mathbf{z}} g(\mathbf{z})\right]$ is defined. Then*

$$
\mathbb{E}_{\mathbf{z}}\left[e^{\theta' \mathbf{z}} g(\mathbf{z})\right] = e^{\frac{\theta' \theta}{2}} \mathbb{E}_{\mathbf{z}}\left[g(\mathbf{z} + \theta)\right]
$$

The next lemma makes use of the following result.

Claim A.0.1 *Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be first-order differentiable functions such that*

[22] For the last equality, since the x_k's are independently distributed, we have $\int_{\mathbb{R}^d} \cdots \int \prod_{k=1}^{d} \frac{\mathrm{d}\mathbf{x}}{\max\{1, x_k^2\}} = \prod_{k=1}^{d} \int_{\mathbb{R}} \frac{\mathrm{d}x_k}{\max\{1, x_k^2\}} = \prod_{k=1}^{d} \left(\int_{-1}^{1} \mathrm{d}x_k + 2 \int_1^\infty \frac{1}{x_k^2} \mathrm{d}x_k \right) = 4^d$

(i) $\int_{\mathbb{R}} \phi(z)\, \psi'(z)\, dz$ *and* $\int_{\mathbb{R}} \phi'(z)\, \psi(z)\, dz$ *are defined, and*

(ii) There exist $m, l \in \mathbb{R}$ *such that* $\int_{-\infty}^{m} \phi(z)\, \psi'(z)\, dz$, $\int_{-\infty}^{m} \phi'(z)\, \psi(z)\, dz$, $\int_{l}^{+\infty} \phi(z)\, \psi'(z)\, dz$, *and* $\int_{l}^{+\infty} \phi'(z)\, \psi(z)\, dz$ *are all defined.*

Then

$$\int_{\mathbb{R}} \phi(z)\, \psi'(z)\, dz = \lim_{d \to +\infty} \phi(d)\, \psi(d) - \lim_{c \to -\infty} \phi(c)\, \psi(c) - \int_{\mathbb{R}} \phi'(z)\, \psi(z)\, dz$$

Proof. For $l, m \in \mathbb{R}$ as in (ii), we can write[23]

$$\int_{\mathbb{R}} \phi(z)\, \psi'(z)\, \mathrm{d}z = \int_{-\infty}^{m} \phi(z)\, \psi'(z)\, \mathrm{d}z + \int_{m}^{l} \phi(z)\, \psi'(z)\, \mathrm{d}z + \int_{l}^{+\infty} \phi(z)\, \psi'(z)\, \mathrm{d}z$$

Using standard integration-by-parts, the proper integral above becomes

$$\int_{m}^{l} \phi(z)\, \psi'(z)\, \mathrm{d}z = \phi(l)\, \psi(l) - \phi(m)\, \psi(m) - \int_{m}^{l} \phi'(z)\, \psi(z)\, \mathrm{d}z$$

The two improper integrals can be written as

$$\begin{aligned} \int_{-\infty}^{m} \phi(z)\, \psi'(z)\, \mathrm{d}z &= \lim_{c \to -\infty} \int_{c}^{m} \phi(z)\, \psi'(z)\, \mathrm{d}z \\ &= \lim_{c \to -\infty} \left(\phi(m)\, \psi(m) - \phi(c)\, \psi(c) - \int_{c}^{m} \phi'(z)\, \psi(z)\, dz \right) \\ &= \phi(m)\, \psi(m) - \lim_{c \to -\infty} \phi(c)\, \psi(c) - \int_{-\infty}^{m} \phi'(z)\, \psi(z)\, \mathrm{d}z \end{aligned}$$

and

$$\begin{aligned} \int_{l}^{+\infty} \phi(z)\, \psi'(z)\, \mathrm{d}z &= \lim_{d \to +\infty} \int_{l}^{d} \phi(z)\, \psi'(z)\, \mathrm{d}z \\ &= \lim_{d \to +\infty} \left(\phi(d)\, \psi(d) - \phi(l)\, \psi(l) - \int_{l}^{d} \phi'(z)\, \psi(z)\, \mathrm{d}z \right) \\ &= \lim_{d \to +\infty} \phi(d)\, \psi(d) - \phi(l)\, \psi(l) - \int_{l}^{+\infty} \phi'(z)\, \psi(z)\, \mathrm{d}z \end{aligned}$$

[23] Given our assumptions, the integrals $\int_{\mathbb{R}} \phi(z)\, \psi'(z)\, \mathrm{d}z$, $\int_{-\infty}^{m} \phi(z)\, \psi'(z)\, \mathrm{d}z$, and $\int_{l}^{+\infty} \phi(z)\, \psi'(z)\, \mathrm{d}z$ are all defined. Clearly, so is the proper integral $\int_{m}^{l} \phi(z)\, \psi'(z)\, \mathrm{d}z$.

Therefore,

$$\begin{aligned}\int_{\mathbb{R}} \phi(z)\,\psi'(z)\,\mathrm{d}z &= \lim_{d\to+\infty}\phi(d)\,\psi(d) - \lim_{c\to-\infty}\phi(c)\,\psi(c) \\ &\quad -\left(\int_{-\infty}^{m}\phi'(z)\,\psi(z)\,\mathrm{d}z + \int_{m}^{l}\phi'(z)\,\psi(z)\,\mathrm{d}z + \int_{l}^{+\infty}\phi'(z)\,\psi(z)\,\mathrm{d}z\right) \\ &= \lim_{d\to+\infty}\phi(d)\,\psi(d) - \lim_{c\to-\infty}\phi(c)\,\psi(c) - \int_{\mathbb{R}}\phi'(z)\,\psi(z)\,\mathrm{d}z\end{aligned}$$

which completes the proof. ■

Lemma A.3 *Let $\mathbf{z}\in\mathbb{R}^d$ be a random vector distributed $\mathbf{N}(\mathbf{0},\mathbf{I}_d)$ and $\theta\in\mathbb{R}^d$. Suppose that $h:\mathbb{R}^d\to\mathbb{R}$ is such that the following conditions are met.*

(i) $\mathbb{E}_{\mathbf{z}}\left[e^{\theta'\mathbf{z}}\frac{\partial h(\mathbf{z})}{\partial z_k}\right]$ *and* $\mathbb{E}_{\mathbf{z}}\left[z_k h(\mathbf{z}+\theta)\right]$ *are defined.*

(ii) For any $\mathbf{z}_{-k}\in\mathbb{R}^{d-1}$, Claim A.0.2 applies on the functions $\psi,\phi:\mathbb{R}\to\mathbb{R}$ given by $\psi(z_k)=h(\mathbf{z})$ and $\phi(z_k)=e^{\theta'\mathbf{z}-\frac{\mathbf{z}'\mathbf{z}}{2}}$.

(iii) $\lim_{z_k\to\pm\infty}\phi(z_k)\,\psi(z_k)=0\ \forall\mathbf{z}_{-k}\in\mathbb{R}^{d-1}$.

Then

$$\mathbb{E}_{\mathbf{z}}\left[e^{\theta'\mathbf{z}}\frac{\partial h(\mathbf{z})}{\partial z_k}\right] = e^{\frac{\theta'\theta}{2}}\mathbb{E}_{\mathbf{z}}\left[z_k h(\mathbf{z}+\theta)\right]$$

Proof. We have

$$\begin{aligned}\mathbb{E}_{\mathbf{z}}\left[e^{\theta'\mathbf{z}}\frac{\partial h(\mathbf{z})}{\partial z_k}\right] &= \underset{\mathbb{R}^d}{\int\cdots\int} e^{\theta'\mathbf{z}}\frac{\partial h(\mathbf{z})}{\partial z_k}\mathrm{d}\Phi(\mathbf{z}) \\ &= \frac{1}{\sqrt{(2\pi)^d}}\underset{\mathbb{R}^{d-1}}{\int\cdots\int}\left(\int_{\mathbb{R}} e^{\theta'\mathbf{z}}\frac{\partial}{\partial z_k}h(\mathbf{z})\,e^{-\frac{z_k^2}{2}}dz_k\right)e^{-\frac{\sum_{i\neq k}z_i^2}{2}}\mathrm{d}\mathbf{z}_{-k}\end{aligned}$$

By Claim A.0.2, we can use integration by parts to simplify the integral in the brackets. Given $\mathbf{z}_{-k}\in\mathbb{R}^{d-1}$, define $\phi:\mathbb{R}\to\mathbb{R}_+$ and $\psi:\mathbb{R}\to\mathbb{R}$ by $\phi(z_k)=e^{\theta'(z_k,\mathbf{z}_{-k})-\frac{z_k^2}{2}}$ and

$\psi(z_k) = h(z_k, \mathbf{z}_{-k})$, respectively. We have

$$\begin{aligned}
& \int_{\mathbb{R}} e^{\theta'\mathbf{z}} \frac{\partial h(\mathbf{z})}{\partial z_k} e^{-\frac{z_k^2}{2}} \mathrm{d}z_k = \int_{\mathbb{R}} \phi(z_k)\, \psi'(z_k)\, \mathrm{d}z_k \\
= \; & \lim_{d \to +\infty} \phi(d)\, \psi(d) - \lim_{c \to -\infty} \phi(c)\, \psi(c) - \int_{\mathbb{R}} \phi'(z)\, \psi(z)\, \mathrm{d}z \\
= \; & \left(\lim_{z_k \to +\infty} e^{\theta'\mathbf{z} - \frac{z_k^2}{2}} g(\mathbf{z}) - \lim_{z_k \to -\infty} e^{\theta'\mathbf{z} - \frac{z_k^2}{2}} g(\mathbf{z}) \right) - \int_{\mathbb{R}} (\theta_k - z_k)\, h(\mathbf{z}) e^{\theta'\mathbf{z} - \frac{z_k^2}{2}} \mathrm{d}z_k \\
= \; & \int_{\mathbb{R}} (z_k - \theta_k)\, h(\mathbf{z}) e^{\theta'\mathbf{z} - \frac{z_k^2}{2}} \mathrm{d}z_k
\end{aligned}$$

Integrating now over $\mathbf{z}_{-k} \in \mathbb{R}^{d-1}$,

$$\begin{aligned}
& \underset{\mathbb{R}^{d-1}}{\int \cdots \int} \left(\int_{\mathbb{R}} e^{\theta'\mathbf{z}} \frac{\partial h(\mathbf{z})}{\partial z_k} e^{-\frac{z_k^2}{2}} \mathrm{d}z_k \right) e^{-\frac{\sum_{i \neq k} z_i^2}{2}} \mathrm{d}\mathbf{z}_{-k} \\
= \; & \underset{\mathbb{R}^{d-1}}{\int \cdots \int} \left(\int_{\mathbb{R}} (z_k - \theta_k)\, h(\mathbf{z}) e^{\theta'\mathbf{z} - \frac{z_k^2}{2}} \mathrm{d}z_k \right) e^{-\frac{\sum_{i \neq k} z_i^2}{2}} \mathrm{d}\mathbf{z}_{-k} \\
= \; & e^{\frac{\theta'\theta}{2}} \underset{\mathbb{R}^{d}}{\int \cdots \int} (z_k - \theta_k)\, h(\mathbf{z}) e^{-\frac{\sum_i (z_i - \theta_i)^2}{2}} \mathrm{d}\mathbf{z} \\
= \; & e^{\frac{\theta'\theta}{2}} \underset{\mathbb{R}^{d}}{\int \cdots \int} (z_k - \theta_k)\, h(\mathbf{z}) e^{-\frac{(\mathbf{z}-\theta)'(\mathbf{z}-\theta)}{2}} \mathrm{d}\mathbf{z} = e^{\frac{\theta'\theta}{2}} \underset{\mathbb{R}^{d}}{\int \cdots \int} z_k h(\mathbf{z} + \theta) e^{-\frac{\mathbf{z}'\mathbf{z}}{2}} \mathrm{d}\mathbf{z}
\end{aligned}$$

The required result follows immediately. ■

Lemma A.4 *For $n \in \mathbb{N} \setminus \{0\}$, let $\mathbf{S} \subseteq \mathbb{R}^n$ be of non-zero Lebesgue measure and such that $\mathbf{S} \times \mathbf{S}$ is symmetric around the origin.*[24] *Suppose that the following conditions are satisfied.*

(i) $g : \mathbf{S} \times \mathbf{S} \to \mathbb{R}_{++}$ *is symmetric,* $g(\mathbf{x}, \mathbf{y}) = g(\mathbf{y}, \mathbf{x})$*, everywhere on its domain except for sets of measure zero.*[25]

(ii) $f : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$ *is such that* $f(\mathbf{x}, \mathbf{y}) + f(\mathbf{y}, \mathbf{x}) \geq 0$ *everywhere on its domain except for sets of measure zero.*

(iii) $gf(\cdot) \equiv g(\cdot)\, f(\cdot)$ *is Lebesgue-integrable over* $\mathbf{S} \times \mathbf{S}$.

[24] By $\mathbf{S} \times \mathbf{S}$ being symmetric around the origin $\mathbf{0} \in \mathbb{R}^{2n}$, we mean that the statement $\langle (\mathbf{x}, \mathbf{y}) \in \mathbf{S} \times \mathbf{S} \Longrightarrow (\mathbf{y}, \mathbf{x}) \in \mathbf{S} \times \mathbf{S} \rangle$ holds for every $(\mathbf{x}, \mathbf{y}) \in \mathbf{S} \times \mathbf{S}$.

[25] More generally, the lemma holds for any $g : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$ that is non-negative and symmetric almost everywhere on its domain as long as $g(\cdot) \neq 0$ on a subset of $\mathbf{S} \times \mathbf{S}$ of positive measure.

Then

$$\int_{\mathbf{S}\times\mathbf{S}}\cdots\int g\left(\mathbf{x},\mathbf{y}\right)f\left(\mathbf{x},\mathbf{y}\right)d\left(\mathbf{x},\mathbf{y}\right)\geq 0$$

The inequality is strict if $f\left(\mathbf{x},\mathbf{y}\right)+f\left(\mathbf{y},\mathbf{x}\right)\neq 0$ *on a subset of* $\mathbf{S}\times\mathbf{S}$ *of non-zero measure.*

Proof. Since $gf\left(\cdot\right)$ is integrable, by the Fubini-Tonelli theorem, the integral in question can be written as an iterated one:

$$\int_{\mathbf{S}\times\mathbf{S}}\cdots\int g\left(\mathbf{x},\mathbf{y}\right)f\left(\mathbf{x},\mathbf{y}\right)\mathrm{d}\left(\mathbf{x},\mathbf{y}\right)=\int_{\mathbf{S}}\cdots\int\left(\int_{\mathbf{S}}\cdots\int g\left(\mathbf{x},\mathbf{y}\right)f\left(\mathbf{x},\mathbf{y}\right)\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x}$$

Simply by re-naming the variables of integration, we also have

$$\begin{aligned}\int_{\mathbf{S}\times\mathbf{S}}\cdots\int g\left(\mathbf{x},\mathbf{y}\right)f\left(\mathbf{x},\mathbf{y}\right)\mathrm{d}\left(\mathbf{x},\mathbf{y}\right) &= \int_{\mathbf{S}\times\mathbf{S}}\cdots\int g\left(\mathbf{y},\mathbf{x}\right)f\left(\mathbf{y},\mathbf{x}\right)\mathrm{d}\left(\mathbf{y},\mathbf{x}\right)\\ &= \int_{\mathbf{S}}\cdots\int\left(\int_{\mathbf{S}}\cdots\int g\left(\mathbf{y},\mathbf{x}\right)f\left(\mathbf{y},\mathbf{x}\right)\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x}\end{aligned}$$

Therefore,

$$\begin{aligned}&2\int_{\mathbf{S}\times\mathbf{S}}\cdots\int g\left(\mathbf{x},\mathbf{y}\right)f\left(\mathbf{x},\mathbf{y}\right)\mathrm{d}\left(\mathbf{x},\mathbf{y}\right)\\ = &\int_{\mathbf{S}}\cdots\int\left(\int_{\mathbf{S}}\cdots\int g\left(\mathbf{x},\mathbf{y}\right)f\left(\mathbf{x},\mathbf{y}\right)\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x}+\int_{\mathbf{S}}\cdots\int\left(\int_{\mathbf{S}}\cdots\int g\left(\mathbf{y},\mathbf{x}\right)f\left(\mathbf{y},\mathbf{x}\right)\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x}\\ = &\int_{\mathbf{S}}\cdots\int\left(\int_{\mathbf{S}}\cdots\int g\left(\mathbf{x},\mathbf{y}\right)\left[f\left(\mathbf{x},\mathbf{y}\right)+f\left(\mathbf{x},\mathbf{y}\right)\right]\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x}\geq 0\end{aligned}$$

The last equality above uses the symmetry of $g\left(\cdot\right)$. The inequality follows from property (ii) of $f\left(\cdot\right)$; it is strict if $f\left(\mathbf{x},\mathbf{y}\right)+f\left(\mathbf{y},\mathbf{x}\right)\neq 0$ on a subset of $\mathbf{S}\times\mathbf{S}$ of positive measure. ■

Lemma A.5 *Let* $\mathbf{x}\in\mathbb{R}^d$, $\mathbf{z}_{-k}\in\mathbb{R}^{d-1}$, *and* $y_k\in\mathbb{R}$ *be random vectors distributed i.i.d.*$\boldsymbol{N}(\mathbf{0},\boldsymbol{I}_d)$, *i.i.d.*$\boldsymbol{N}(\mathbf{0},\boldsymbol{I}_{d-1})$, *and* $\boldsymbol{N}(0,1)$, *respectively. Consider, moreover,* $\phi:\mathbf{R}^d\to\mathbb{R}_{++}$ *such that*

$\mathbb{E}_{\mathbf{x}}\left[\phi\left(\mathbf{x}\right)\right]$ *is defined and let* $\psi:\mathbb{R}\rightarrow\mathbb{R}$ *be given by*

$$\psi\left(y_k\right)=\mathbb{E}_{\mathbf{x}}\left[\phi\left(\mathbf{x}\right)\left(y_k-x_k\right)\right]\mathbb{E}_{\mathbf{z}_{-k}}\left[\phi\left(\mathbf{z}_{-k},y_k\right)\right]$$

There exists $\lambda_k\in\mathbb{R}$*:* $\quad \left(y_k-\lambda_k\right)\psi\left(y_k\right)>0 \quad \forall y_k\in\mathbb{R}\setminus\{\lambda_k\}$.

Proof. Observe that

$$\begin{aligned}\psi\left(y_k\right) &= \left(\mathbb{E}_{\mathbf{x}}\left[\phi\left(\mathbf{x}\right)\right]y_k-\mathbb{E}_{\mathbf{x}}\left[\phi\left(\mathbf{x}\right)x_k\right]\right)\mathbb{E}_{\mathbf{z_{-k}}}\left[\phi\left(\mathbf{z}_{-k},y_k\right)\right]\\ &= \mathbb{E}_{\mathbf{x}}\left[\phi\left(\mathbf{x}\right)\right]\left(y_k-\frac{\mathbb{E}_{\mathbf{x}}\left[\phi\left(\mathbf{x}\right)x_k\right]}{\mathbb{E}_{\mathbf{x}}\left[\phi\left(\mathbf{x}\right)\right]}\right)\mathbb{E}_{\mathbf{z_{-k}}}\left[\phi\left(\mathbf{z}_{-k},y_k\right)\right]\end{aligned}$$

Since $\mathbb{E}_{\mathbf{x}}\left[\phi\left(\mathbf{x}\right)\right]$ and $\mathbb{E}_{\mathbf{z}_{-k}}\left[\phi\left(\mathbf{z}_{-k},y_k\right)\right]$ are strictly positive quantities for any $y_k\in\mathbb{R}$, it suffices to define $\lambda_k=\frac{\mathbb{E}_{\mathbf{x}}\left[\phi(\mathbf{x})x_k\right]}{\mathbb{E}_{\mathbf{x}}\left[\phi(\mathbf{x})\right]}$. ■

B Comonotonicity and Covariance

Borrowing from the Appendix of Chateauneuf *et al.* [14], this section presents the relevant (for my argument in Section 2) part of the proof to their claim. Notice that the random variables in question need not be bounded in my application. The boundedness condition guarantees the existence of the relevant expectations for *any* additive probability measure. In the text, we focus attention on the M-dimensional Normal probability distribution and take $g:\mathbb{R}^M\rightarrow\mathbb{R}_-$ and $f:\mathbb{R}^M\rightarrow\mathbb{R}_+$ given by $f\left(\mathbf{x}_M\right)=e^{\mu_j T+\sigma_j\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}_M\right)}$ and $g\left(\mathbf{x}_M\right)=\varphi_2''\left(F_{-M}\left(\omega,t,\mathbf{x}_{-M}\right)+f\left(\mathbf{x}_M\right)\right)$. The relevant expectations are well-defined even though f,g are not and not necessarily, respectively, bounded.

Definition B.1 *For a set* $\mathbf{S}$ *and an algebra* σ *on* $\mathbf{S}$*, let* $\left(\mathbf{S},\sigma\right)$ *be a probabilisable space and* $B\left(\mathbf{S},\mathbb{R}\right)$ *be the set of bounded* σ*-measurable functions* $\mathbf{S}\mapsto\mathbb{R}$*. Two random variables* $g,f\in B\left(\mathbf{S},\mathbb{R}\right)$ *are* ***comonotonic*** *if*

$$\left[g\left(\mathbf{x}\right)-g\left(\mathbf{y}\right)\right]\left[f\left(\mathbf{x}\right)-f\left(\mathbf{y}\right)\right]\geq 0 \quad \forall \mathbf{x},\mathbf{y}\in\mathbf{S}:\mathbf{x}\neq\mathbf{y}$$

Claim B.0.2 $g,f\in B\left(\mathbf{S},\mathbb{R}\right)$ *are comonotonic if and only if* $Cov_\mu\left[g,f\right]\geq 0$ *for all additive probability measures* μ *on* $\left(\mathbf{S},\sigma\right)$*.*

Proof. (Only if) Suppose that

$$[g(\mathbf{x}) - g(\mathbf{y})][f(\mathbf{x}) - f(\mathbf{y})] \geq 0 \quad \forall \mathbf{x}, \mathbf{y} \in \mathbf{S} : \mathbf{x} \neq \mathbf{y}$$

and let μ be an additive probability measure on $(\mathbf{S}, \sigma)$. We have

$$\begin{aligned}
& 2\mathrm{Cov}_{\mu}[g, f] = 2\left(\mathbb{E}_{\mu}[gf] - \mathbb{E}_{\mu}[g]\,\mathbb{E}_{\mu}[f]\right) \\
= & \int_{\mathbf{S}} g(\mathbf{x}) f(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x}) + \int_{\mathbf{S}} g(\mathbf{y}) f(\mathbf{y})\,\mathrm{d}\mu(\mathbf{y}) \\
& - \int_{\mathbf{S}} g(\mathbf{y})\,\mathrm{d}\mu(\mathbf{y}) \int_{\mathbf{S}} f(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x}) - \int_{\mathbf{S}} g(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x}) \int_{\mathbf{S}} f(\mathbf{y})\,\mathrm{d}\mu(\mathbf{y}) \\
= & \int_{\mathbf{S}\times\mathbf{S}} [g(\mathbf{x}) - g(\mathbf{y})][f(\mathbf{x}) - f(\mathbf{y})]\,\mathrm{d}\mu(\mathbf{x})\,\mathrm{d}\mu(\mathbf{y}) \geq 0 \qquad \square
\end{aligned}$$

Since non-increasing absolute risk aversion ($r'_A(\cdot) \leq 0$) implies $\varphi_2'''(\cdot) > 0$, the claim applies.