University of Texas at El Paso

From the SelectedWorks of Kien H Lim

July 2007

Improving students' algebraic thinking: The case of Talia

Contact Author

Start Your Own SelectedWorks

Notify Me of New Work

Available at: http://works.bepress.com/kien_lim/2

Lim, K. H. (2007). Improving students' algebraic thinking: The case of Talia. In J. Woo, H. Lew, K. Park, & D. Seo (Eds.), *Proceedings of the Thirty-first Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 193-200). Seoul, Korea: The Korea Society of Educational Studies in Mathematics.

IMPROVING STUDENTS' ALGEBRAIC THINKING: THE CASE OF TALIA

Kien Lim

The University of Texas at El Paso

This paper presents the case of an 11th grader, Talia, who demonstrated improvement in her algebraic thinking after five one-hour sessions of solving problems involving inequalities and equations. She improved from association-based to coordination-based predictions, from impulsive to analytic anticipations, and from inequality-as-a-signal-for-a-procedure to inequality-as-a-comparison-of-functions conceptions. In the one-on-one teaching intervention, she progressed from the sub-context of manipulating symbols, to working with specific numbers, to reasoning with "general" numbers, and eventually to reasoning with symbols. Three features were identified to account for her improvement: (a) attention to meaning, (b) opportunity to repeat similar reasoning, and (c) opportunity to explore.

INTRODUCTION

Research has shown that some students will spontaneously apply a procedure or algorithm as soon as they are given a mathematics problem. For example, Cramer, Post, and Currier (1993) observed 32 of 33 students in a mathematics methods class apply the proportion algorithm to solve this problem: *Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?* Without appearing to understand the underlying structures, many students inappropriately apply procedures taught to them; Fischbein and Barash (1993) call this *improper application of algorithmic models*. Errors such as thinking that $\frac{3a+5b}{2a+5c} = \frac{1}{2} \Rightarrow \frac{3+b}{2+c} = \frac{1}{2}$ and $(x-6)(x-9) < 0 \Rightarrow x < 6$ or $x < 9$ are rather common among algebra students (Matz, 1980). Many students fail to make the connection between *structural conception* and *operational conception* (Sfard, 1991). For example, they interpret an equation as an object to be transformed into "x = ___". The only source of meaning is the rules for solving the equation (Sfard & Linchevski, 1994). Without conceiving an equation as a relation, high school and college students may interpret the equal sign as a signal to do something—for example, to solve for a variable or to find its derivative (Kieran, 1981). Consequently, students exhibit *non-referential symbolic reasoning* (Harel, in press) when they operate on symbols as if "the symbols possess a life of their own" without attending to referential meaning.

The research that this paper reports sought to characterize the way students *anticipate* as they solve non-routine problems involving algebraic inequalities and equations. I define *anticipating* as a mental act of conceiving a certain expectation without performing a sequence of detailed operations to arrive at the expectation. This

research had three objectives: (a) to identify and characterize students' anticipations, (b) to identify the relationship between the characteristics of students' anticipations and students' interpretation of inequalities/equations, and (c) to explore the potential for advancing the way students anticipate. Results related to the first two objectives were presented in the 28th PME-NA conference (Lim, 2006). This paper presents results related to the third objective.

THEORETICAL FRAMEWORK

This research was based on several theoretical constructs: Piaget's (1967/1971) notion of anticipation, von Glasersfeld's (1998) three general kinds of anticipation, and Cobb's (1985) three hierarchical levels of anticipation. In addition, Harel's (in press) notions of way of understanding and way of thinking were employed to analyse students' act of anticipating.

According to Piaget (1967/1971), anticipation is one of the two functions of knowing; the other function being conservation-of-information, an instrument of which is a scheme. The anticipation function deals with the application of a scheme to a new situation. It allows us to strategize and plan, have foresight, make predictions, formulate conjectures, engage in thought experiments, etc. Foresights and predictions are possible because of our ability to assimilate situations into our existing scheme(s); "anticipation is nothing other than a transfer or application of the scheme … to a new situation before it actually happens" (p. 195).

von Glasersfeld (1998) elaborated on Piaget's notion of anticipation by pointing to three general kinds of anticipation: (a) implicit expectations that are present in our actions, e.g., the preparation and control of our movements when we grope in the dark; (b) prediction of an outcome, e.g., predicting that it will soon rain upon noticing that the sky is being covered by dark clouds; and (c) foresight of a desired event and the means for attaining it, e.g., a child's anticipation of the capitulation of his parent if he were to throw a temper tantrum in public. In this research, I focused on the latter two kinds of anticipation. I define *predicting* as the act of conceiving an expectation for the result of an event without actually performing the operations associated with the event, and *foreseeing* as the act of conceiving an expectation that leads to an action, prior to performing the operations associated with the action.

Cobb (1985) identifies three hierarchical levels of anticipation: beliefs, problem-solving heuristics, and conceptual structures. At the global level, students' beliefs about mathematics influence their anticipations. At the intermediate level, a child anticipates a *heuristic*—"a metacognitive prompt which delimits a subcontext within which the child anticipates she can elaborate and solve the problem" (Cobb, 1985, p. 124). For example, anticipation of a guess-and-check strategy may lead a student to operate in the sub-context of plugging in numbers. At the most specific level, a child's expressed conceptual structures (i.e., evoked schemes) dictate the child's anticipations. According to Cobb, higher-level anticipations constrain lower-level

anticipations, i.e., students' specific anticipations are confined both by their beliefs and by the sub-context in which they operate.

I used Harel's (in press) MA-WoU-WoT framework to analyse students' *mental acts* (MA_s) of predicting and foreseeing. Predicting and foreseeing are among the many mental acts that one might carry out while solving a mathematics problem. Other mental acts include interpreting, symbolizing, generalizing, justifying, and inferring. A *way of understanding* (WoU) refers to the product of a particular mental act, and a *way of thinking* (WoT) refers to a character of this act. Taking the act of predicting as an example, a WoU refers to the result a student actually predicts whereas a WoT characterizes the manner in which the student predicts. Likewise, students' interpretations of inequalities/equations can be viewed as W_soU associated with the act of interpreting inequalities/equations.

METHOD

This research was conducted in a university-based charter school in Southern California. Fourteen 11th graders were interviewed, each for about 60 minutes. Four of these interviewees participated in a one-on-one teaching intervention, which involved five problem-solving sessions followed by a post-interview.

Tasks used in the clinical interviews include: (a) Is there a value for x that will make $(2x - 6)(x - 3) < 0$ true? (b) Given that $5a = b + 5$, which is larger: a or b? And (c) Given that m is greater than n, can $m - 14$ ever be equal to $7 - n$? These tasks differ from typical tasks in textbooks in that they do not direct students to perform a specific task such as "solve for x" or "simplify." I found this non-directive feature effective at eliciting a variety of anticipatory behaviours. All the tasks were phrased in the form of a question so that students could predict the answer, if they chose to, prior to performing any actions. Tasks used in the teaching intervention involved only one variable. This way, participants' responses to two-variable tasks in the post-interview allowed me to see whether the improvements in their W_soT and W_soU went beyond the context in which these W_soT and W_soU were learned.

The designing, sequencing, and assigning of tasks in the teaching intervention were guided by the three primary pedagogical principles in Harel's DNR-based instruction (2001, in press). The *Duality Principle* asserts that the W_soT students possess influence the W_soU they produce, which in turn influences the development of their W_soT. The *Necessity Principle* stipulates that for students to learn a particular concept, they must have an intellectual need for it. The *Repeated-reasoning Principle* asserts that "students must practice reasoning in order to internalize, organize, and retain" what they learn.

All the interviews and problem-solving sessions were videotaped and transcribed. *Observation concepts* (Clement, 2000) for students' W_soT associated with predicting/foreseeing and students' W_soU inequalities/equations were identified. Categories for W_soT and W_soU were derived from the data using a *constant*

comparative approach (Glaser & Strauss, 1967), in which categories were constantly revised by comparing current data with previously analysed data. The analysis involved identifying instances of the mental acts of predicting and foreseeing (inferred from student's actions and statements), generating, comparing, and refining categories for W_soT and W_soU, and consolidating and collapsing some of the categories. For each of the four learners, a table of codes was created to track the changes from the pre-interview to the post-interview in: (a) the learner's W_soT associated with predicting/foreseeing, (b) W_soU inequalities/equations, (c) *sub-context* in which the learner was operating, and (d) quality/correctness of solutions. Episodes of all five problem-solving sessions for the learner Talia were analysed to gain a general sense of her ways of thinking and ways of understanding. I later revisited the data to account for significant transitions as well as to account for the change in her ways of thinking and ways of understanding.

RESULTS AND DISCUSSION

In this study, three ways of thinking associated with predicting were identified: *association-based prediction*, *comparison-based prediction*, and *coordination-based prediction*. Five ways of thinking associated with foreseeing were identified: *impulsive anticipation*, *tenacious anticipation*, *explorative anticipation*, *analytic anticipation*, and *interiorized anticipation*. In addition, five ways of understanding inequalities/equations (I/E) were identified: *I/E-as-a-signal-for-procedure*, *I/E-as-a-static-comparison*, *I/E-as-a-proposition*, *I/E-as-a-constraint*, and *I/E-as-a-comparison-of-functions*. Students' W_soT associated with predicting/foreseeing were found to be related to the quality of their solutions as well as to the sophistication of their W_soU inequalities/equations. These results are presented in PME-NA (Lim, 2006). In this paper, I focus on Talia's improvement from pre-interview to post-interview, her trajectory from the sub-context of manipulating symbols to the sub-context of reasoning with symbols, and some features of the teaching intervention that might account for her improvement.

Talia's Pre-to-post-interview Improvement

In the pre-interview, Talia was operating in the *sub-context of manipulating symbols* for single-variable tasks. While operating in this sub-context, she tended to be procedure-oriented and thus exhibited *impulsive anticipation*. For the task, "Is there a value for x that will make the following statement true? $(2x - 6)(x - 3) < 0$", she spontaneously expanded the expression without studying the inequality, used the quadratic formula, obtained $\frac{6\pm\sqrt{6^2-4(1)(9)}}{2}$, and commented "that reduces to 3, which is less than 0 (wrote $3 < 0$). That's not true." Her not attending to the meaning of the symbols contributed to her exhibiting *association-based prediction*. She predicted that 3 was not a solution because she saw $3 < 0$ was false. Her prediction was based on her associating the result of $\frac{6\pm\sqrt{6^2-4(1)(9)}}{2}$ with the output of $x^2 - 6x + 9$; i.e., she conflated the root of a quadratic function with the function itself. When she plugged

3 into the inequality and obtained 0 < 0, she predicted that 6 might be a solution: "Maybe I'm supposed to multiply by 2." Because of her association between the value of $\frac{6\pm\sqrt{6^2-4(1)(9)}}{2}$ and the function $x^2 - 6x + 9$, she thought she should double the resulting value of 3 so as to compensate for halving $2(x^2 - 6x + 9)$ to get $x^2 - 6x + 9$.

In the post-interview, Talia was operating in the *sub-context of reasoning with symbols*. While operating in this sub-context, she tended to be goal-oriented, and thus exhibited *analytic anticipation*.

> Talia: Um, $2x$ minus 6 times x minus 3 is less than 0. So … this [side] has to give me a negative number. I can get a negative number from here $(2x - 6)$, oh, but there is also a negative times negative is positive. So I have to make one of these negative and one of these positive. In order to get this, so this will be negative if it is less than 6, but then if I want to make this one positive, it has to be greater than 3. So, or I could go the other way around. … This side could be, umm, greater than 6, x could be greater than 6, makes this positive, $2x$, I'm sorry, $2x$ [could be greater than 6]. And x could be less than 3, which will make this negative, and so these two conditions will make this statement true.

Talia analysed the inequality with the goal of making the function $(2x - 6)(x - 3)$ less than zero, and she foresaw the sub-goal of making one factor positive and the other negative. Talia's pre-to-post-interview improvement, as depicted in Figure 6.1, is considered significant because only 2 out of the 16 inequalities/equations used in the teaching intervention involved quadratic functions in factored form. Moreover, both inequalities, $x(6x + 8) < 0$ and $3x(500 - 2x) < 30(500 - 2x)$, do not involve repeated roots.

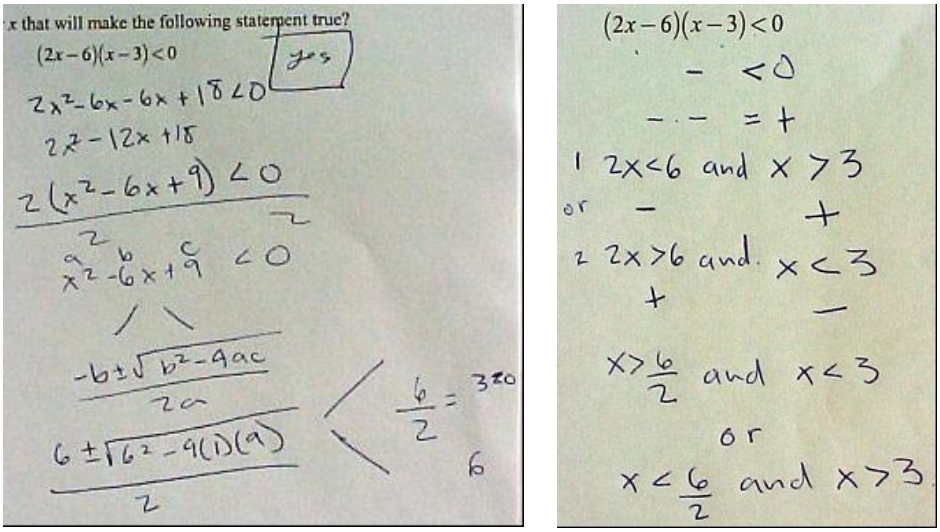

Figure 6.1: Pre-and-post-interview comparison of Talia's work

When working on two-variable tasks, Talia demonstrated more instances of coordination-based prediction and analytic anticipation in the post-interview than in the pre-interview. For example, she exhibited only one instance of comparison-based prediction in the pre-interview, but two instances of coordination-based prediction and one instance of comparison-based prediction in the post-interview for this task:

"Given that $5a = b + 5$, which is larger: a or b?" In the pre-interview, her prediction appeared to be based on a comparison between the two sides in terms of their arithmetic operations: "if a and b were equal, then a would be larger because, I mean this ($5a$) value would be larger." In the post-interview, her prediction, though still incorrect, incorporates change and compensation: "b will have to be larger, just because you need more adding than you do multiplying in order to get [$b + 5$] large."

Talia's Trajectory from Manipulating-symbols to Reasoning-with-Symbols

Talia's transition from the sub-context of manipulating symbols to the sub-context of reasoning with numbers involved two intermediate stages: working with specific numbers and reasoning with general numbers (e.g., large positive numbers, small positive numbers, and negative numbers). In the first problem-solving session, when Talia was presented with the inequality $\frac{x-5}{x-10} < 0$ with no accompanying instruction, she interpreted the inequality as a signal to solve for x.

Lim: Alright, this ($\frac{x-5}{x-10} < 0$) is the first problem.

Talia: OK. So I just solve it? Alright, arrr, so I'm trying to solve for x. So I'm just going to multiply both sides by x minus 10, x minus 10, and it's x minus 5 is less than 0. And then you just add 5 to both sides. x is less than 5. Um, I think that's my answer.

Lim: What does this answer ($x < 5$) mean?

Talia: Um, that, this equation is true for any value of x that are less than 5, so, let me just try that out. So, 4 minus 5 over 4 minus 10.

Having found that $x = 4$ did not make the inequality true, Talia continued to think of alternative means for manipulating the inequality: "How am I supposed to solve this? Um, maybe I can factor something out." It was only when she was asked, "What does solve for x mean?" that she attended to meaning and responded, "To find the values for this problem where the statement is true." She then foresaw plugging in numbers.

Talia: So, umm, I'm just going to try some random values for this, 2. 2 minus 5 is -3. 2 minus 10 [is] -8. Umm, it has to be a number that is positive on the top and negative on the bottom, so I can get a negative number, and then the statement will be true. So, something that will give me positive is 6 minus 5, and 6 minus 10. This is…positive 1, over negative, um, 10, 3 4 5 6 (finger counting), 4 and that's less than 0. So one-fourth is a value that makes this statement true. … I'm sorry, 6.

Within the context of working with specific numbers, Talia could reason in a goal-oriented manner and foresaw plugging in 6 to make the numerator positive and the denominator negative. She even extended her reasoning to obtain all the values that would make the inequality true: "So x can be anything that is, um, bigger than 5, but less than 10. So 6 7, 6 7 8, 9."

The change in sub-context from manipulating symbols to plugging in numbers was probably initiated by questions such as "What does this answer mean?" and "What does solve for x mean?" This implies that mathematics teachers should help students *attend to meaning*.

The transition from working with specific numbers to reasoning with general numbers occurred in Talia's initial response to the second task: "Is $x(6x + 8) < 0$ always true, sometimes true, or never true?"

> Talia: Is x [times] quantity of $6x$ plus 8 less than 0 always true, sometimes true, or never true? Mmm, I'm thinking if I make x into a negative number so that, um, so that the whole function will be negative. So if there is an answer, it will probably have to be negative because if I make x positive, it's going to be greater than 0 all the time. Right? … OK. Um, so let me just try a negative number, -1.

In this task, Talia began to reason with general numbers: x being positive would make the inequality "greater than 0 all the time." The transition from working with specific numbers to working with general numbers might be due to the inequality having x as a factor. An implication for teaching is that instructional tasks should be designed to allow students to apply, and then extend, their ways of understanding. The quadratic inequality $x(6x + 8) < 0$ is considered a good follow-up to the rational inequality $\frac{x-5}{x-10} < 0$ because the two functions are structurally different, yet they both foster the same way of thinking: being goal-oriented so as to make one factor positive and one factor negative. Hence, assigning $x(6x + 8) < 0$ as the second task is consistent with the *Repeated-reasoning Principle* (Harel, 2001).

The transition from reasoning with general numbers to reasoning with symbols began with the above task and continued through the entire teaching intervention. Reasoning with symbols involves certain ways of understanding, emergence of which required Talia to explore the problem situations by plugging in specific numbers and/or reasoning with general numbers, such as a number in the interval [-1, 0]. This observation suggests that mathematics educators should use reasoning with numbers as a platform for students to explore algebraic structures. I contend that Talia's undesirable ways of thinking—such as non-referential symbolic way of thinking and association-based prediction—probably resulted from her working with algebraic symbols without the support of numbers. A lack of numerical support for algebraic reasoning is a plausible cause for why some students perceive the world of algebra and the world of arithmetic to be disconnected, a phenomenon observed by Lee and Wheeler (1989).

CONCLUSION

The case of Talia demonstrates the feasibility of helping students improve their algebraic thinking—in particular, moving from manipulating symbols in a non-referential symbolic manner to reasoning with symbols in a goal-oriented manner, from association-based prediction to coordination-based prediction, and from impulsive anticipation to analytic anticipation. This research underscores the importance of helping students attend to meaning, creating opportunities for students to repeat certain reasoning, and using numbers as a platform for students to investigate algebraic expressions and structures.

References

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547-589). Mahwah, NJ: Lawrence Erlbaum Associates.

Cobb, P. (1985). Two children's anticipation, beliefs and motivations. *Educational Studies in Mathematics, 16* (2), 111-126.

Cramer, K., Post, T., & Currier, S. (1993). Learning and teaching ratio and proportion: Research implications. In D. T. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 159-178). NY: Macmillan.

Fishbein, E., & Barash, A. (1993). Algorithmic models and their misuse in solving algebraic problems. In I. Hirabayashi, N. Nohda, K. Shigematsu & F. Lin (Eds.), *Proc. 17th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 162-172). Tsukuba, Japan: PME.

Glaser, B., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago: Aldine.

Harel, G. (2001). The development of mathematical induction as a proof scheme: A model for DNR-based instruction. In S. Campbell & R. Zaskis (Eds.), *Learning and teaching number theory: Research in cognition and instruction* (pp. 185-211). Westport, CT: Ablex.

Harel, G. (in press). The DNR system as a conceptual framework for curriculum development and instruction. In R. Lesh, J. Kaput, E. Hamilton & J. Zawojewski. *Foundations for the future: The need for new mathematical understandings & abilities in the 21st century*. Hillsdale, NJ: Lawrence Erlbaum.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*, p. 317-326.

Lee, L., & Wheeler, D. (1989). The arithmetic connection. *Educational Studies in Mathematics, 20*, 41-54.

Lim, K. (2006). Characterizing students' thinking: Algebraic inequalities and equations. In S. Alatorre, J. L. Cortina, M. Sáiz & A. Méndez (Eds.), *Proc. 28th Annual Meeting of the North American Chapter of the Int. Group for the Psychology of Mathematics Education*, (Vol. 2, pp. 102-109). Mérida, México: PME-NA.

Matz, M. (1980). Towards a computational theory of algebraic competence. *Journal of Mathematical Behavior, 3*, 93-166.

Piaget, J. (1971). *Biology and knowledge* (B. Walsh, Trans.). Chicago: University of Chicago Press. (Original work published 1967)

Sfard, A., & Linchevski, L. (1994). Between arithmetic and algebra: In the search of a missing link the case of equations and inequalities. *Rendiconti Del Seminario Matematico, 52*, 279-307.

von Glasersfeld, E. (1998). Anticipation in the constructivist theory of cognition. In D.M.Dubois (Ed.) *Computing anticipatory systems* (pp. 38-47), Woodbury, NY: American Institute of Physics.