Multiboard determinacy

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Miami University, Oxford, OH, November 20, 2009

Games

In set theory we study certain games, formalized as follows: We fix a set X; the elements of X are the possible moves.

Two players, I and II, alternate, with I playing first. The rules of each particular game determine which moves are valid. In each turn, the corresponding player picks an element of X that is a valid move; if there is no such element, the player loses. If the game continues for infinitely many moves, the winner is decided depending on the sequence in $X^{\mathbb{N}}$ that has been produced.

Both players know the rules of the game, have access to the elements of X and, at each turn, know what moves have been made so far. We call these *perfect information* games.

Finite games

I	x_0		x_2		$\dots$
II		x_1		x_3	

If X is finite and any run of the game always ends after a finite number of moves, we say that the game is *finite*.

Typical examples are tic-tac-toe and chess. Games of chance do not fall within this framework.

Another restriction is that games cannot end in a tie. This would eliminate chess from our analysis, but we can include it if we decree that, say, black "wins" if the game ends up tied.

Finite games

Formally, a *strategy* is a function that assigns to each finite sequence of elements of X an element of X,

$$\sigma : X^{<\mathbb{N}} \to X.$$

We say that I *follows* the strategy σ if each move of I is dictated by σ and the previous moves of II:

I	$\sigma(\langle\rangle)$		$\sigma(\langle x_1\rangle)$		$\sigma(\langle x_1, x_3\rangle)$		
II		x_1		x_3			$\dots$

Similarly, we can talk of II *following* σ.

Finite games

We say that σ is a *winning strategy* for I in a game $\mathfrak{G}$ if I wins whenever I follows σ. (Similarly for II.) A game is *determined* if one of the players has a winning strategy.

The mathematical analysis of games began with Zermelo (1913), with chess being the game originally singled out. From Zermelo's arguments it follows that all finite games are determined.

Bengt Ekeroth and Max von Sydow in *The Seventh Seal*, 1957. © Svensk Filmindustri; Ingmar Bergman (director), Gunnar Fischer (cinematography).

(Aside)

There is a version of *paper-rock-scissors* as a finite perfect information game between three players. Of course, this game is undetermined.

The moral is that if we are interested in determinacy, we must limit our analysis to two-player games.

Infinite games

Gale and Stewart noticed that the argument showing the determinacy of finite games readily generalizes to show that open games are determined.

A game $\mathfrak{Y}$ is *open* for player I if, whenever I wins, this had already been decided after finitely many moves.

Given $A \subseteq X^{\mathbb{N}}$, the game $\mathfrak{Y}_X(A)$ is defined so that there are no restrictions on what elements of X can be played at each turn, and I wins iff the resulting sequence $x = \langle x_0, x_1, \dots \rangle$ is an element of A.

Infinite games

Theorem (Gale-Stewart, 1953)

Suppose X is given the discrete topology, and $X^{\mathbb{N}}$ the product topology. If A is an open subset of $X^{\mathbb{N}}$, then $\mathfrak{D}_X(A)$ is determined.

A careful inductive argument generalized this as follows:

Theorem (Martin, 1975)

If A is a Borel subset of $X^{\mathbb{N}}$, then $\mathfrak{D}_X(A)$ is determined.

Infinite games

Using the theory of large cardinals, one can extend the analysis when $X = \mathbb{N}$ much further:

Theorem (Martin-Steel, Woodin, 1985)

If $A \in L(\mathbb{R})$, then $\mathfrak{Dn}(A)$ is determined.

The class $L(\mathbb{R})$ is a model of set theory without choice; it contains every subset of $\mathbb{N}^{\mathbb{N}} \simeq \mathbb{R} \setminus \mathbb{Q}$ that is reasonably definable. Certainly, every Borel set, and every set that ever appears in Analysis.

Infinite games

On the other hand, we cannot hope that every game is determined:

Fact

There is a set $A \subseteq \mathbb{N}^{\mathbb{N}}$ with $\mathfrak{Y}_{\mathbb{N}}(A)$ undetermined.

The *Axiom of determinacy*, suggested by Mycielski and Steinhaus in 1962, states that all games $\mathfrak{Y}_{\mathbb{N}}(A)$ are determined. The fact above indicates that this axiom is inconsistent with the usual system of axioms of set theory, ZFC. The result of Martin-Steel and Woodin mentioned before indicates that it is consistent with ZF, the usual system excluding the axiom of choice.

(Aside)

In our framework, it is obviously impossible that both players have winning strategies.

In a sense, this is due to the discrete nature of the game. If we imagine a run of the game taking place in time, players do not move in tandem. In a continuous game we would imagine time as modeled by $[0,1]$ rather than $\mathbb{N}$, and the object a player produces would be a function $f : [0,1] \to X$ perhaps with some additional continuity requirements. Say I is playing f_I and II is playing f_{II}. For any $t \in [0,1]$, to choose $f_I(t)$, player I has access to $f_{II} \restriction [0,t)$ and similarly with I and II exchanged.

Bollobás, Leader, and Walters have recently shown examples of games of this kind (lion-man pursuit and evasion games) where *both* players have winning strategies.

Multiboard games

In the 1970s, Fred Galvin suggested an approach to the study of undetermined games $\mathfrak{Gd}_X(A)$. One fixes a cardinal κ. In the game $\mathfrak{Gd}_X(A, \kappa)$, rather than playing in one board, players I and II compete simultaneously on κ many boards. Player I wins iff there is some board in which I wins the corresponding run of $\mathfrak{Gd}_X(A)$.

Multiboard determinacy

Given $A \subseteq X^{\mathbb{N}}$ there are three possibilities:

- Player II has a winning strategy in $\mathfrak{Y}_X(A)$ and therefore in $\mathfrak{Y}_X(A, \kappa)$ for all κ.
- There is some κ_0 such that $\mathfrak{Y}_X(A, \lambda)$ is undetermined if $\lambda < \kappa_0$ but player I has a winning strategy if $\lambda \geq \kappa_0$.
- $\mathfrak{Y}_X(A, \kappa)$ is undetermined for all κ.

Multiboard determinacy

Galvin obtained results and formulated questions about these games, but they remain unpublished and mostly unknown.

A survey is (finally) being prepared in joint work with Galvin, Ketchersid, and Scheepers.

In what follows I want to present some of the results we have, and the most interesting problems that remain.

Consistency

Perhaps somewhat surprisingly, given that the determinacy of all games $\mathfrak{Dn}_{\mathbb{N}}(A)$ contradicts choice and that, even without choice, there are undetermined games $\mathfrak{Dn}_{\omega_1}(A)$, we have:

Theorem

It is consistent that for all X and all $A \subseteq X^{\mathbb{N}}$, there is some κ such that $\mathfrak{Dn}_X(A, \kappa)$ is determined.

Here is a high level sketch of the argument:

The cut-and-choose game

In the *cut-and-choose game* $\mathsf{CC}(\lambda, \kappa)$, players NON-EMPTY and EMPTY alternate as follows:

First, NON-EMPTY chooses a partition of κ into λ pieces, and then EMPTY chooses one of these, S_0.

Then EMPTY chooses a partition of S_0 into λ pieces, and NON-EMPTY chooses one of these, S_1.

Then NON-EMPTY chooses a partition of S_1 into λ pieces, and EMPTY chooses one of these, say S_2, and so on.

NON-EMPTY wins if $\bigcap_{n\in\mathbb{N}} S_n \neq \emptyset$. Else, EMPTY wins.

The cut-and-choose game

Theorem

NON-EMPTY has a winning strategy in $\mathsf{CC}(2, \kappa)$ iff NON-EMPTY has a winning strategy in $\mathsf{CC}(\omega, \kappa)$, and that in this case $\kappa > |\mathbb{R}|$.

Theorem

If NON-EMPTY has a winning strategy in $\mathsf{CC}(\lambda, \kappa)$, then all games $\mathfrak{Y}_\lambda(A, \kappa)$ are determined.

It then suffices to show that it is consistent that the cut-and-choose games are determined.

Ideals

Given a set X, a collection $\mathcal{I} \subseteq \mathcal{P}(X)$ is a *nonprincipal ideal* iff:

- $\{a\} \in \mathcal{I}$ for all $a \in X$.
- $X \notin \mathcal{I}$.
- If $A \subseteq B \in \mathcal{I}$, then $A \in \mathcal{I}$.
- If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

We can think of nonprincipal ideals as an abstraction of the collection of null sets with respect to some complete non-atomic measure. Let $\mathcal{I}^+$ denote the collection of "$\mathcal{I}$-positive" sets, those that meet the complement of every set in $\mathcal{I}$.

A property of ideals

Given infinite cardinals κ and λ, the property $\mathsf{I}(\lambda, \kappa)$ holds iff there is a nonprincipal ideal $\mathcal{I} \subseteq \mathcal{P}(\kappa)$ such that:

- Whenever $(A_\alpha \mid \alpha \in \lambda)$ is a collection of sets in $\mathcal{I}$, then also $\bigcup_\alpha A_\alpha \in \mathcal{I}$;
- Each $X \in \mathcal{I}^+$ has a partition into λ disjoint sets, each in $\mathcal{I}^+$;
- There is a family $\mathcal{F} \subseteq \mathcal{I}^+$ such that:
  1. Every $X \in \mathcal{I}^+$ contains a $Y \in \mathcal{F}$, and
  2. Every descending ω-sequence of sets from $\mathcal{F}$ has a nonempty intersection.

A property of ideals

The following is immediate from the definitions:

Lemma

For all infinite cardinals κ and λ, if $\mathsf{I}(\lambda, \kappa)$, then NON-EMPTY has a winning strategy in $\mathsf{CC}(\lambda, \kappa)$.

Hence, the following completes the proof:

Theorem (Laver)

It is consistent that for all λ, $2^\lambda = \lambda^+$ and $\mathsf{I}(\lambda, \lambda^{++})$.

Conjectures and questions

Conjecture

The following holds (not just consistently): For every X there is a κ such that all games $\mathfrak{Y}_X(A, \kappa)$ are determined.

We do not even know whether the conjecture holds in the presence of large cardinals.

Conjectures and questions

In the argument sketched above, one obtains a model where CH holds, and for all $A \subseteq \mathbb{N}^{\mathbb{N}}$, the game $\mathfrak{Ɔ}_{\mathbb{N}}(A, \omega_2)$ holds. Moreover, our approach cannot give us anything better, because it depends on NON-EMPTY winning $\mathrm{CC}(\mathbb{N}, \kappa)$, and this requires that $\kappa > |\mathbb{R}|$.

Question

Is there a set $A \subseteq \mathbb{N}^{\mathbb{N}}$ such that $\mathfrak{Ɔ}_{\mathbb{N}}(A, |\mathbb{R}|)$ is undetermined?

Theorem

There is a set $A \subseteq \mathbb{N}^{\mathbb{N}}$ such that for all $\kappa < |\mathbb{R}|$, the game $\mathfrak{Ɔ}_{\mathbb{N}}(A, \kappa)$ is undetermined.

Conjectures and questions

Given $\kappa < |\mathbb{R}|$, we are far from obtaining a precise calibration of what sets $A \subseteq \mathbb{N}^{\mathbb{N}}$ are such that $\mathfrak{Ɔ}_{\mathbb{N}}(A, \kappa)$ is undetermined.

Theorem

For all $n \leq |\mathbb{N}|$ there is a set $A_n \subseteq \mathbb{N}^{\mathbb{N}}$ such that $\mathfrak{Ɔ}_{\mathbb{N}}(A_n, m)$ is undetermined for all $m < n$, and player I has a winning strategy in $\mathfrak{Ɔ}_{\mathbb{N}}(A_n, n)$ that ensures that, in any run of the game, II wins in fewer than n boards.

Conjectures and questions

Let's call *κ-determinacy* the statement that all games $\mathfrak{D}_{\mathbb{N}}(A, \kappa)$ are determined. If $\kappa < |\mathbb{R}|$, this statement is inconsistent with choice, but follows from the usual axiom of determinacy.

Theorem

In the theory $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$, ω-determinacy implies the axiom of determinacy in $L(\mathbb{R})$.

Question

Does ω-determinacy imply determinacy?

Question

Is ω_1-determinacy consistent with choice?

Consistency strength

The result above shows that ω-determinacy and determinacy are equiconsistent. In terms of large cardinals, they correspond to the existence of ω Woodin cardinals.

Laver's result mentioned above gives us that $|\mathbb{R}|^+$-determinacy only requires in consistency strength a measurable cardinal. But we do not know whether *any* large cardinals are needed.

Now I want to mention some topological problems that can help us test the conjecture above:

Powerfully Baire spaces

Recall that a topological space is *Baire* iff the intersection of countably many dense open sets is dense.

Say that a space X is *productively Baire* iff all the powers X^κ are Baire.

Theorem (Fleissner-Kunen)

A space X is productively Baire iff $X^{\mathbb{N}}$ is Baire.

Powerfully Baire spaces

Recall the *Banach-Mazur game* on a space X. In this game, EMPTY and NON-EMPTY take turns playing nonempty open sets U_i with $U_{i+1} \subseteq U_i$, $i \in \mathbb{N}$. EMPTY starts the game by choosing U_0.

NON-EMPTY wins a play of the game if the intersection $\bigcap_{i \in \mathbb{N}} U_i$ is nonempty. Else, EMPTY wins.

Theorem (Banach)

EMPTY has a winning strategy in the Banach-Mazur game on X iff X is not Baire.

X is said to be *Choquet* if NON-EMPTY has a winning strategy in the Banach-Mazur game on X. If X is a Choquet space, then all (finite or infinite) powers of X in the box topology, and thus the Tychonoff topology, are Choquet and in particular Baire.

Powerfully Baire spaces

Say that X is *powerfully Baire* iff all box powers X^κ are Baire.

Fact

Suppose that for all λ there is a κ such that all games $\mathfrak{D}_\lambda(A, \kappa)$ are determined.
Then a space is Choquet iff it is powerfully Baire.

Powerfully Baire spaces

Our first test problem:

Conjecture

A space X is powerfully Baire iff X is Choquet.

Our understanding of powerfully Baire spaces is far form satisfactory. For example:

Question

Suppose that X and Y are powerfully Baire. Is $X \times Y$ powerfully Baire? Is X productively Baire?

(Note the conjecture immediately answers the question positively.)

Paracompactness

Recall that a space X is *paracompact* iff every open cover admits an open locally finite refinement, i.e., given any open cover $\mathcal{U}$ of X, there is an open cover $\mathcal{V}$ of X such that:

1. $\mathcal{V}$ is a refinement: Each $V \in \mathcal{V}$ is contained in some member of $\mathcal{U}$, and
2. $\mathcal{V}$ is locally finite: For every point $x \in X$ there is an open neighborhood W of x that only meets finitely many of the members of $\mathcal{V}$.

Paracompactness

It has long been an objective of set theoretic topologists to determine which non-compact locally compact spaces are paracompact.

From now on we assume that X is a non-compact but completely regular topological space.

A family $\mathcal{F}$ of compact subsets of X is a *moving off family* iff $\mathcal{F}$ is nonempty, each member of $\mathcal{F}$ is a nonempty compact subset of X, and for each compact subset C of X there is an $F \in \mathcal{F}$ such that $C \cap F = \emptyset$.

Paracompactness

A family $\mathcal{S}$ of nonempty subsets of X is *discrete* iff, for each $x \in X$, there is an open set U such that $x \in U$ and

$$|\{S \in \mathcal{S} \colon U \cap S \neq \emptyset\}| \leq 1.$$

A family of sets $\mathcal{S}$ has a *discrete open expansion* iff, for each $S \in \mathcal{S}$, there is an open set U_S such that $S \subseteq U_S$, and $\{U_S \colon S \in \mathcal{S}\}$ is a discrete family of subsets of X.

Paracompactness

The space X has the *moving off property* if each moving off family of X has an infinite subset which has a discrete open expansion.

Theorem (Gruenhage)

If a locally compact T_2-space is paracompact, then it has the the moving off property.

(The converse fails in general, but holds for locally compact ordered spaces.)

Question

Is there an appropriate strengthening of the moving off property equivalent to paracompactness in the class of locally compact T_2-spaces?

Paracompactness

Recall that the compact-open topology on $C(X)$ is determined by the subbase consisting of sets of the form

$$V(K,U) = \{f \in C(X) \colon f[K] \subseteq U\}$$

where K is a compact subset of X and U is a finite open interval of $\mathbb{R}$.

Let $C_k(X)$ denote $C(X)$ equipped with this topology.

Paracompactness

Theorem (Gruenhage-Ma)

For a locally compact T_2-space the following are equivalent:

- $C_k(X)$ is a Baire space.
- X has the moving off property.

This easily implies the following result:

Paracompactness

Theorem (Ma)

If X is a locally compact T_2-space, then $C_k(X)$ is Choquet iff X is paracompact.

This leads to our second test problem. Our previous conjecture that powerfully Baire spaces are Choquet immediately implies:

Conjecture

Let X be a locally compact T_2-space. Then $C_k(X)$ is powerfully Baire iff X is paracompact.

(Stolen from http://worldrecordsacademy.org)

The end.