

## SOME ASYMPTOTIC METHODS FOR STRONGLY NONLINEAR EQUATIONS

JI-HUAN HE

*College of ience, Donghua University,  
 Shanghai 200051, People's Republic of China  
 jhhe@dhu.edu.cn*

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This paper features a survey of some recent developments in asymptotic techniques, which are valid not only for weakly nonlinear equations, but also for strongly ones. Further, the obtained approximate analytical solutions are valid for the whole solution domain. The limitations of traditional perturbation methods are illustrated, various modified perturbation techniques are proposed, and some mathematical tools such as variational theory, homotopy technology, and iteration technique are introduced to overcome the shortcomings.

In this paper the following categories of asymptotic methods are emphasized: (1) variational approaches, (2) parameter-expanding methods, (3) parameterized perturbation method, (4) homotopy perturbation method (5) iteration perturbation method, and ancient Chinese methods.

The emphasis of this article is put mainly on the developments in this field in China so the references, therefore, are not exhaustive.

*Keywords:* Analytical solution; nonlinear equation; perturbation method; variational theory; variational iteration method; homotopy perturbation; ancient Chinese mathematics.

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## 1. Introduction

With the rapid development of nonlinear science, there appears an ever-increasing interest of scientists and engineers in the analytical asymptotic techniques for nonlinear problems. Though it is very easy for us now to find the solutions of linear systems by means of computer, it is, however, still very difficult to solve nonlinear problems either numerically or theoretically. This is possibly due to the fact that the various discredited methods or numerical simulations apply iteration techniques to find their numerical solutions of nonlinear problems, and nearly all iterative methods are sensitive to initial solutions,<sup>14,27,40,81,101,102,107</sup> so it is very difficult to obtain converged results in cases of strong nonlinearity. In addition, the most important information, such as the natural circular frequency of a nonlinear oscillation depends on the initial conditions (i.e. amplitude of oscillation) will be lost during the procedure of numerical simulation.

Perturbation methods<sup>8,11,15,18,22,94,98–100</sup> provide the most versatile tools available in nonlinear analysis of engineering problems, and they are constantly being developed and applied to ever more complex problems. But, like other nonlinear asymptotic techniques, perturbation methods have their own limitations:

- (1) Almost all perturbation methods are based on such an assumption that a small parameter must exist in an equation. This so-called small parameter assumption greatly restricts applications of perturbation techniques, as is well known, an overwhelming majority of nonlinear problems, especially those having strong nonlinearity, have no small parameters at all.
- (2) It is even more difficult to determine the so-called small parameter, which seems to be a special art requiring special techniques. An appropriate choice of small parameter may lead to ideal results, however, an unsuitable choice of small parameter results in badly effects, sometimes serious ones.
- (3) Even if a suitable small parameter exists, the approximate solutions solved by the perturbation methods are valid, in most cases, only for the small values of the parameter. For example, the approximations solved by the method of multiple scales are uniformly valid as long as a specific system parameter is small. However, we cannot rely fully on the approximations because there is no criterion on how small the parameters should be. Thus checking numerically and/or experimentally the validity of the approximations is essential.<sup>83</sup> For example, we consider the following two equations:<sup>113</sup>

$$u' + u - \varepsilon = 0, \quad u(0) = 1 \tag{1.1}$$

and

$$u' - u - \varepsilon = 0, \quad u(0) = 1. \quad (1.2)$$

The solution of Eq. (1.1) is  $u(t) = \varepsilon + (1 - \varepsilon)e^{-t}$ , while the unperturbed solution of Eq. (1.1) is  $u_0(t) = e^{-t}$ . In case  $\varepsilon \ll 1$ , the perturbation solution leads to high accuracy, for  $|u(t) - u_0(t)| \leq \varepsilon$ .

By similar analysis for Eq. (1.2), we have

$$|u(t) - u_0(t)| = \varepsilon|1 - e^t|. \quad (1.3)$$

It is obvious that  $u_0$  can never be considered as an approximate solution of Eq. (1.2) in case  $t \gg 1$ , so the traditional perturbation method is not valid for this case even when  $\varepsilon \ll 1$ .

In what follows, we should introduce some new developed methods for problem solving in areas where traditional techniques have been unsuccessful.

There exist some alternative analytical asymptotic approaches, such as the non-perturbative method,<sup>33</sup> weighted linearization method,<sup>10,105</sup> Adomian decomposition method,<sup>1,9,35,118,119</sup> modified Lindstedt–Poincaré method,<sup>28,65–67,91</sup> variational iteration method,<sup>49,70,74–77</sup> energy balance methods,<sup>63</sup> tanh-method,<sup>5,13,36</sup>  $F$ -expansion method<sup>106,115,116</sup> and so on. Just recently, some new perturbation methods such as artificial parameter method,<sup>69,88</sup>  $\delta$ -method,<sup>11,16</sup> perturbation incremental method,<sup>26</sup> homotopy perturbation method,<sup>44–48,50–52,62,72</sup> parameterized perturbation method,<sup>73</sup> and bookkeeping artificial parameter perturbation method<sup>64</sup> which does not depend on on the small parameter assumption are proposed. A recent study<sup>68</sup> also reveals that the numerical technique can also be powerfully applied to the perturbation method. A review of perturbation techniques in phase change heat transfer can be found in Ref. 12, and a review of recently developed analytical techniques can be found in Ref. 71. Recent advance in nonlinear dynamics, vibrations and control in China can be found in Ref. 93.

A wide body of literature dealing with the problem of approximate solutions to nonlinear equations with various different methodologies also exists. Although a comprehensive review of asymptotic methods for nonlinear problems would be in order at this time, this paper does not undertake such a task. Our emphasis is mainly put on the recent developments of this field in China, hence, our references to literature will not be exhaustive. Rather, our purpose in this paper is to present a comprehensive review of the past and current work on some new developed asymptotic techniques for solving nonlinear problems, with the goal of setting some ideas and improvements which point toward new and interesting applications with these methods. The content of this paper is partially speculative and heuristic, but we feel that there is enough theoretical and computational evidence to warrant further investigation and most certainly, refinements and improvements, should the reader choose to pursue the ideas.

In this review article, we limit ourselves to nonlinear order differential equations, most nonlinear partial differential equations can be converted into order differential

ones by some linear translation, for example, the KdV equation

$$u_t + uu_x + \varepsilon u_{xxx} = 0. \quad (1.4)$$

If the wave solution is solved, we can assume that  $u(x, t) = u(\xi) = u(x - ct)$ , where  $c$  is the wave speed. Substituting the above traveling wave solution into Eq. (1.4), we have

$$-cu_\xi + uu_\xi + \varepsilon u_{\xi\xi\xi} = 0. \quad (1.5)$$

Integrating Eq. (1.5), we have an ordinary differential equation:

$$-cu + \frac{1}{2}u^2 + \varepsilon u_{\xi\xi} = D, \quad (1.6)$$

where  $D$  is an integral constant.

In Sec. 2, variational approaches to soliton solution, bifurcation, limit cycle, and period solutions of nonlinear equations are illustrated, including the Ritz method, energy method, variational iteration method; In Sec. 3, we first introduce some modifications of Lindstedt–Poincaré method, then parameter-expanding methods are illustrated where a parameter (or a constant) is expanded into a series in  $\varepsilon$  which can be a small parameter in the equation, or an artificial parameter, or a book-keeping parameter; Sec. 4 introduces parametrized perturbation method where the expanding parameter is introduced by a linear transformation. Homotopy perturbation method is systemically illustrated in Sec. 5. In Sec. 6, the iteration technology is introduced to the perturbation method; Sec. 7 briefly introduces some ancient Chinese methods and their modern applications are illustrated. Finally, conclusions are made in Sec. 8.

## 2. Variational Approaches

### 2.1. Ritz method

Variational methods<sup>38,55–57,89,90,121</sup> such as Raleigh–Ritz and Bubnov–Galerkin techniques have been, and continue to be, popular tools for nonlinear analysis. When contrasted with other approximate analytical methods, variational methods combine the following two advantages:

- (1) They provide physical insight into the nature of the solution of the problem. For example, we consider the statistically unstable Duffing oscillator, defined by

$$u'' - u + \varepsilon u^3 = 0. \quad (2.1)$$

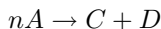
Its variational formulations can be obtained with ease

$$J(u) = \int \left\{ -\frac{1}{2}u'^2 - \frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4 \right\} dt. \quad (2.2)$$

If we want to search for a period solution for Eq. (2.1), then from the energy integral (2.2) it requires that the potential  $-\frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4$  must be positive for all  $t > 0$ , so

an oscillation about the origin will occur only if  $\varepsilon A^2 > 2$ , where  $A$  is the amplitude of the oscillation.

- (2) The obtained solutions are the best among all the possible trial-functions. As an illustration, consider the following chemical reaction<sup>92</sup>



which obeys the equation

$$\frac{dx}{dt} = k(a - x)^n, \tag{2.3}$$

where  $a$  is the number of molecules  $A$  at  $t = 0$ , and  $x$  is the number of molecules  $C$  (or  $D$ ) after time  $t$ , and  $k$  is a reaction constant. At the start of reaction ( $t = 0$ ), there are no molecules  $C$  (or  $D$ ) yet formed so that the initial condition is  $x(0) = 0$ .

In order to obtain a variational model, we differentiate both sides of Eq. (2.3) with respect to time, resulting in

$$\frac{d^2x}{dt^2} = -kn(a - x)^{n-1} \frac{dx}{dt}. \tag{2.4}$$

Substituting Eq. (2.3) into Eq. (2.4), we obtain the following 2-order differential equation

$$\frac{d^2x}{dt^2} = -k^2n(a - x)^{2n-1}. \tag{2.5}$$

The variational model for the differential equation (2.5) can be obtained with ease by the semi-inverse method,<sup>55</sup> which reads

$$J(x) = \int_0^\infty \left\{ \frac{1}{2} \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} k^2 (a - x)^{2n} \right\} dt. \tag{2.6}$$

It is easy to prove that the stationary condition of the above functional, Eq. (2.4), is equivalent to Eq. (2.3).

We apply the Ritz method to obtain an analytical solution of the discussed problem. When all molecules  $A$  are used up, no further increase in the number of molecules  $C$  can occur, so  $dx/dt = 0$ . Putting this into Eq. (2.3), the final number of molecules  $C$  is  $x_\infty = a$ . By the above analysis, we choose a trial-function in the form

$$x = a(1 - e^{-\eta t}) \tag{2.7}$$

where  $\eta$  is an unknown constant to be further determined.

It is obvious that the function (2.7) satisfies the initial condition. Substituting the function (2.7) into Eq. (2.6), we obtain

$$J(\eta) = \int_0^\infty \left\{ \frac{1}{2} a^2 \eta^2 e^{-2\eta t} + \frac{1}{2} k^2 a^{2n} e^{-2n\eta t} \right\} dt = \frac{1}{4} a^2 \eta + \frac{1}{4n\eta} k^2 a^{2n}. \tag{2.8}$$

Minimizing the functional, Eq. (2.6), with respect to  $x$ , is approximately equivalent to minimizing the above function, Eq. (2.8), with respect to  $\eta$ . We set

$$\frac{\partial J}{\partial \eta} = \frac{1}{4}a^2 - \frac{1}{4n\eta^2}k^2a^{2n} = 0 \quad (2.9)$$

which leads to the result

$$\eta = \frac{1}{\sqrt{n}}ka^{n-1}. \quad (2.10)$$

So we obtain an explicit analytical solution for the discussed problem

$$x = a[1 - \exp(-n^{-1/2}ka^{n-1}t)]. \quad (2.11)$$

The unknown constant,  $\eta$ , in the function (2.7) can be identified by various other methods, for example various residual methods,<sup>38</sup> but the result (2.10) is the most optimal one among all other possible choices.

The solution (2.11) states that  $x$  increases approximately exponentially from 0 to a final value  $x = a$ , with a time constant given by the result (2.10). Chemists and technologists always want to know its halfway through the change, i.e.  $x = a/2$  when  $t = t_{1/2}$ . An approximate halfway time obtained from the problem (2.11) reads

$$t_{1/2} = -\frac{1}{\eta} \ln 0.5 = -\frac{1}{n^{-1/2}ka^{n-1}} \ln 0.5. \quad (2.12)$$

To compare with the exact solution, we consider the case when  $n = 2$ . Under such a case, we can easily obtain the following exact solution:

$$x_{\text{exact}} = a \left( 1 - \frac{1}{1 - akt} \right). \quad (2.13)$$

Its halfway time reads

$$\tilde{t}_{1/2} = -\frac{1}{ak}. \quad (2.14)$$

From the half-way time (2.12) we have

$$t_{1/2} = -\frac{1}{2^{-1/2}ka} \ln 0.5 = -\frac{0.98}{ak}. \quad (2.15)$$

The 2% accuracy is remarkable good. Further improvements in accuracy can be achieved by suitable choice of trial-functions, and are outside the purpose of the paper.

The above technology can be successfully applied to many first order nonlinear differential equations arising in physics. Now we consider the Riccati equation<sup>117</sup> in the form

$$y' + p(x)y + q(x)y^2 + r(x) = 0. \quad (2.16)$$

Differentiating Eq. (2.16) with respect to  $x$ , and eliminating  $y'$  in the resulted equation, Eq. (2.16) becomes

$$y'' + (p' + p^2 + 2qr)y + (q' + pq + 2pq)y^2 + 2q^2y^3 + r' + pr = 0. \quad (2.17)$$

Its variational formulation can be expressed as

$$J(y) = \int \left\{ -\frac{1}{2}y'^2 + \frac{1}{2}(p' + p^2 + 2qr)y^2 + \frac{1}{3}(q' + pq + 2pq)y^3 + \frac{1}{2}q^2y^4 + (r' + pr)y \right\} dx. \tag{2.18}$$

Variational approach can also be applied to simplify nonlinear equation, consider the Lambert equation:

$$y'' + \frac{k^2}{n}y = (1 - n)\frac{y'^2}{y}. \tag{2.19}$$

By the semi-inverse method,<sup>55</sup> its variational formulation reads

$$J(y) = \int \left\{ -\frac{1}{2}n^2y^{2n-2}y'^2 + \frac{1}{2}k^2y^{2n} \right\} dx. \tag{2.20}$$

Hinted by the above energy form, Eq. (2.20), we introduce a transformation:

$$z = y^n. \tag{2.21}$$

Functional (2.20) becomes

$$J(z) = \int \left\{ -\frac{1}{2}z'^2 + \frac{1}{2}k^2z^2 \right\} dx. \tag{2.22}$$

Its Euler–Lagrange equation is

$$z'' + k^2z = 0, \tag{2.23}$$

which is a linear equation. The solution of Eq. (2.19), therefore, has the form:

$$y = (C \cos kx + D \sin kx)^{1/n}. \tag{2.24}$$

### 2.2. Soliton solution

There exist various approaches to the search for soliton solution for nonlinear wave equations.<sup>19–21,39</sup> Here, we suggest a novel and effective method by means of variational technology. As an illustrative example, we consider the KdV equation

$$\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{2.25}$$

We seek its traveling wave solutions in the following frame

$$u(x, t) = u(\xi), \quad \xi = x - ct, \tag{2.26}$$

Substituting the solution (2.26) into Eq. (2.25) yields

$$-cu' - 6uu' + u''' = 0, \tag{2.27}$$

where prime denotes the differential with respect to  $\xi$ .

Integrating Eq. (2.27) yields the result

$$-cu - 3u^2 + u'' = 0. \tag{2.28}$$

By the semi-inverse method,<sup>55</sup> the following variational formulation is established

$$J = \int_0^\infty \left( \frac{1}{2}cu^2 + u^3 + \frac{1}{2} \left( \frac{du}{d\xi} \right)^2 \right) d\xi. \tag{2.29}$$

By Ritz method, we search for a solitary wave solution in the form

$$u = p \operatorname{sech}^2(q\xi), \tag{2.30}$$

where  $p$  and  $q$  are constants to be further determined.

Substituting Eq. (2.30) into Eq. (2.29) results in

$$\begin{aligned} J &= \int_0^\infty \left[ \frac{1}{2}cp^2 \operatorname{sech}^4(q\xi) + p^3 \operatorname{sech}^6(q\xi) + \frac{1}{2}4p^2q^2 \operatorname{sech}^4(q\xi) \tanh^2(q\xi) \right] d\xi \\ &= \frac{cp^2}{2q} \int_0^\infty \operatorname{sech}^4(z) dz + \frac{p^3}{q} \int_0^\infty \operatorname{sech}^6(z) dz + 2p^2q \int_0^\infty \{ \operatorname{sech}^4(z) \tanh^4(z) \} dz \\ &= \frac{cp^2}{3q} + \frac{8p^3}{15q} + \frac{4p^2q}{15}. \end{aligned} \tag{2.31}$$

Making  $J$  stationary with respect to  $p$  and  $q$  results in

$$\frac{\partial J}{\partial p} = \frac{2cp}{3q} + \frac{24p^2}{15q} + \frac{8pq}{15} = 0, \tag{2.32}$$

$$\frac{\partial J}{\partial q} = -\frac{cp^2}{3q^2} - \frac{8p^3}{15q^2} + \frac{4p^2}{15} = 0, \tag{2.33}$$

or simplifying

$$5c + 12p + 4q^2 = 0, \tag{2.34}$$

$$-5c - 8p + 4q^2 = 0. \tag{2.35}$$

From Eqs. (2.34) and (2.35), we can easily obtain the following relations:

$$p = -\frac{1}{2}c, \quad q = \sqrt{\frac{c}{4}}. \tag{2.36}$$

So the solitary wave solution can be approximated as

$$u = -\frac{c}{2} \operatorname{sech}^2 \sqrt{\frac{c}{4}}(x - ct - \xi_0), \tag{2.37}$$

which is the exact solitary wave solution of KdV equation (2.25).

**2.3. Bifurcation**

Bifurcation arises in various nonlinear problems.<sup>45,47,48,79,122–125</sup> Variational method can also be applied to the search for dual solutions of a nonlinear equation, and to approximate identification of the bifurcation point. Consider the Bratu’s equation in the form:

$$u'' + \lambda e^u = 0, \quad u(0) = u(1) = 0, \tag{2.38}$$

where  $\lambda$  is a positive constant called as the eigenvalue. Equation (2.38) comes originally from a simplification of the solid fuel ignition model in thermal combustion theory,<sup>42</sup> and it has an exact solution

$$u(x) = -2 \log \left[ \cosh \left( \frac{0.5(x - 0.5)\theta}{\cosh(\theta/4)} \right) \right],$$

where  $\theta$  solves

$$\theta = \sqrt{2\lambda} \cosh(\theta/4).$$

Due to its importance, the Bratu problem has been studied extensively. First, it arises in a wide variety of physical applications, ranging from chemical reaction theory, radiative heat transfer and nanotechnology<sup>114</sup> to the expansion of universe.<sup>42</sup> Second, because of its simplicity, the equation is widely used as a benchmarking tool for numerical methods. Khuri<sup>42</sup> applied Adomian decomposition method to study the problem, the procedure is tedious though the expression is a lovely closed-form, furthermore only one solution is found when  $0 < \lambda < \lambda_c$ , and error arises when  $\lambda \rightarrow \lambda_c$ .

The present problem can be easily solved with high accuracy by the variational method, furthermore, two branches of the solutions can be simultaneously determined, and the bifurcation point can be identified with ease. It is easy to establish a variational formulation for Eq. (2.38), which reads

$$J(u) = \int_0^1 \left\{ \frac{1}{2} u'^2 - \lambda e^u \right\} dx. \tag{2.39}$$

In view of the Ritz method, we choose the following trial-function:

$$u = Ax(1 - x), \tag{2.40}$$

where  $A$  is an unknown constant to be further determined. Substituting the function (2.40) into the formula (2.39) results in:

$$J(A) = \int_0^1 \left\{ \frac{1}{2} A^2 (1 - 2x)^2 - \lambda e^{Ax(1-x)} \right\} dx. \tag{2.41}$$

Minimizing the formula (2.41) to determine the constant, this yields

$$\frac{\partial J}{\partial A} = \int_0^1 \{ A(1 - 2x)^2 - \lambda x(1 - x)e^{Ax(1-x)} \} dx = 0. \tag{2.42}$$

Equation (2.42) can be readily solved by some mathematical software, for example, MATHEMATICA, Matlab. There are two solutions to Eq. (2.42) for values of  $0 < \lambda < \lambda_c$ , and no solutions for  $\lambda > \lambda_c$ . There is only one solution for a critical value of  $\lambda = \lambda_c$ , which solves

$$\int_0^1 \{(1 - 2x)^2 - \lambda_c x^2(1 - x)^2 e^{A_c x(1-x)}\} dx = 0. \tag{2.43}$$

Dividing Eq. (2.43) by Eq. (2.42) produces

$$A_c = \frac{\int_0^1 \{x(1 - x)e^{A_c x(1-x)}\} dx}{\int_0^1 \{x^2(1 - x)^2 e^{A_c x(1-x)}\} dx}. \tag{2.44}$$

Solving Eq. (2.44), we have  $A_c = 4.72772$ . Substituting the result into Eq. (2.43) results in  $\lambda_c = 3.56908$ . The exact critical value is  $\tilde{\lambda}_c = 3.51383071$ . The 1.57% accuracy is remarkable good in view of the crude trial-function, Eq. (2.40).

In order to compare with Khuri’s results, we consider the case  $\lambda = 1$ . In such case, Eq. (2.43) becomes

$$\int_0^1 \{A(1 - 2x)^2 - x(1 - x)e^{Ax(1-x)}\} dx = 0. \tag{2.45}$$

Solving this equation by Mathematica:

```
Find Root [Integrate [A*(1 - 2*x)^2 - x*(1 - x)Exp[A*x*(1 - x)], {x, 0, 1}]
= 0, {A, 1}],
```

we have  $A_1 = 0.559441$ , and  $A_2 = 16.0395$ .

In the case  $\lambda = 1$ , the Bratu equation has two solutions  $u_1$  and  $u_2$  with  $u'_1(0) = 0.549$  and  $u'_2(0) = 10.909$ . Our solutions predict that  $u'_1(0) = 0.559441$ , and  $u'_2(0) = 16.0395$ , with accuracies of 1.9% and 46.94%, respectively. Accuracy can be improved if we choose a more suitable trial-function, for example,  $u = Ax(1 - x)(1 + Bx + Cx^2)$ .

### 2.4. Limit cycle

Energy approach (Variational approach) to limit cycle is suggested by He,<sup>58,59</sup> which can be applied to not only weakly nonlinear equations, but also strongly ones, and the obtained results are valid for whole solution domain. D’Acunto<sup>30</sup> called the energy approach as He’s variational method.

To illustrate the basic idea of the method, we consider the following nonlinear oscillation:

$$\ddot{x} + x + \varepsilon f(x, \dot{x}) = 0, \tag{2.46}$$

In the present study the parameter  $\varepsilon$  need not be small.

Generally speaking, limit cycles can be determined approximately in the form

$$x = b + a(t) \cos \omega t + \sum_{n=1}^m (C_n \cos n\omega t + D_n \sin n\omega t), \tag{2.47}$$

where  $b$ ,  $C_n$  and  $D_n$  are constant.

Substituting Eq. (2.47) into Eq. (2.46) results in the following residual

$$R(t) = \ddot{x} + x + \varepsilon f(x, \dot{x}). \tag{2.48}$$

In general, the residual might not be vanishingly small at all points, the error depends upon the infinite “work”,  $dw$ , done by the “force”  $R$  in the infinite distance  $dx$ :

$$dw = Rdx. \tag{2.49}$$

We hope the totally work done in a period is zero, that requires

$$\int_{A_0}^{A_1} Rdx = 0, \tag{2.50}$$

where  $A_0$  and  $A_1$  are minimum and maximum amplitudes, respectively.

Equation (2.50) can be equivalently written in the form

$$\int_0^T R\dot{x}dt = 0, \tag{2.51}$$

where  $T$  is the period, this technique is similar to the method of weighted residuals.

We assume that the amplitude weakly varies with time, so we write the amplitude in the form

$$a(t) = Ae^{\alpha t} \approx A, \quad |\alpha| \ll 1. \tag{2.52}$$

Accordingly we *approximately* have the following expressions

$$\dot{x} \approx \alpha x - \omega \sqrt{A^2 - x^2}, \tag{2.53}$$

$$\ddot{x} \approx (\alpha^2 - \omega^2)x - 2\alpha\omega \sqrt{A^2 - x^2}. \tag{2.54}$$

Now we consider the van der Pol equation:

$$\ddot{x} + x - \varepsilon(1 - x^2)\dot{x} = 0, \tag{2.55}$$

We will not re-illustrate the solution procedure, but we identify  $\alpha$  in Ref. 58 instead with

$$\alpha = \frac{\varepsilon^2(A^2 - 4)\omega\pi}{4(2\pi\omega - \varepsilon A^2)}, \tag{2.56}$$

and the frequency equation (33) in Ref. 58 should be replaced by

$$1 - \omega^2 + \frac{\varepsilon^2(A^2 - 4)^2\omega^2\pi^2}{16(2\pi\omega - \varepsilon A^2)^2} - \frac{\varepsilon^2(A^2 - 4)\omega\pi}{4(2\pi\omega - \varepsilon A^2)} = 0. \tag{2.57}$$

Now consider the limit  $\varepsilon \ll 1$ , by assumption  $\omega = 1 + a\varepsilon^2 + O(\varepsilon^3)$ , Eq. (2.57) becomes

$$\left[-2a + \frac{\varepsilon^2(A^2 - 4)^2}{64} + \frac{(A^2 - 4)}{8}\right] \varepsilon^2 + O(\varepsilon^3) = 0. \tag{2.58}$$

After identification the constant,  $a$ , we have

$$\omega = 1 + \varepsilon^2 \left[\frac{(A^2 - 4)^2}{128} + \frac{A^2 - 4}{16}\right] + O(\varepsilon^3). \tag{2.59}$$

In the case  $\varepsilon \rightarrow \infty$ , the frequency can be approximated as

$$\omega = \frac{b}{\varepsilon} + O\left(\frac{1}{\varepsilon^2}\right), \tag{2.60}$$

From Eq. (2.57), we have

$$1 - \left(\frac{b}{\varepsilon^2}\right) + \frac{b^2(A^2 - 4)^2 \pi^2}{16(2\pi b/\varepsilon - \varepsilon A^2)^2} - \frac{\varepsilon b(A^2 - 4)\pi}{4(2\pi b/\varepsilon - \varepsilon A^2)} + O\left(\frac{1}{\varepsilon^2}\right) = 0. \tag{2.61}$$

If we keep the order of  $1/\varepsilon$ , then Eq. (2.61) can be simplified as

$$1 + \frac{b(A^2 - 4)\pi}{4A^2} = 0, \tag{2.62}$$

from which the constant,  $b$ , can be readily determined. As a result, we have:

$$\omega = -\frac{4A^2}{(A^2 - 4)\pi\varepsilon} + O\left(\frac{1}{\varepsilon^2}\right). \tag{2.63}$$

If we choose  $A = 2$  which is identified by perturbation method,<sup>104</sup> then the last two terms in Eq. (2.57) vanish completely, leading to the invalidity of the method as pointed out by Rajendran *et al.*<sup>104</sup> The perturbation method provides us with a choice of the approximate value of  $A$ , and many alternative approaches to the identification of  $A$  exist.

Assuming that period solution has the form:  $x = A \cos \omega t$ , we obtain the following residual:

$$R = (1 - \omega^2)A \cos \omega t + A\varepsilon\omega(1 - A^2 \cos^2 \omega t) \sin \omega t. \tag{2.64}$$

By substituting Eq. (2.59) or Eq. (2.63) into Eq. (2.64) respectively, we have

$$R = A\varepsilon(1 - A^2 \cos^2 \omega t) \sin \omega t + O(\varepsilon^2), \quad \varepsilon \rightarrow 0, \tag{2.65}$$

and

$$R/\varepsilon = A(1 - A^2 \cos^2 \omega t) \sin \omega t + O(1/\varepsilon), \quad 1/\varepsilon \rightarrow 0. \tag{2.66}$$

The value of  $A$  can be determined in the parlance of vanishing small residual by the method of weighted residuals.

Applying the Galerkin method

$$\int_0^{T/4} R \sin \omega t dt = \int_0^{T/4} A(1 - A^2 \cos^2 \omega t) \sin^2 \omega t dt = 0, \quad T = 2\pi/\omega, \tag{2.67}$$

leading to the result:  $A = 2$ , which agrees with the standard result of the perturbation method.

The simplest method among the method of weighted residuals is the method of collocation. Collocating at  $\omega t = \pi/3$ ,  $\omega t = \pi/4$ , and  $\omega t = \pi/6$  gives, respectively, the approximate values of  $A$ :  $A = 2$ ,  $A = 1.414$ , and  $A = 1.155$ . The average value of  $A$  reads  $A = 1.522$ .

We can also apply the least squares method to find the approximate value of  $A$  by the formulation:

$$\int_0^{T/4} R \frac{\partial R}{\partial A} dt = 0, \tag{2.68}$$

i.e.

$$\int_0^{T/4} A(1 - A^2 \cos^2 \omega t)(1 - 3A^2 \cos^2 \omega t) \sin^2 \omega t dt = 0, \tag{2.69}$$

which leads to the result  $A = \sqrt{8/3} = 1.633$ .

Now we choose  $A = 1.633$ , then Eq. (2.59) becomes

$$\omega = 1 - \frac{5}{72}\varepsilon^2 + O(\varepsilon^3) = 1 - 0.0694\varepsilon^2 + O(\varepsilon^3). \tag{2.70}$$

We write down the perturbation solution for reference, which reads

$$\omega = 1 - \frac{1}{16}\varepsilon^2 + O(\varepsilon^3) = 1 - 0.0625\varepsilon^2 + O(\varepsilon^3). \tag{2.71}$$

Equation (2.63) becomes

$$\omega = \frac{8}{\pi\varepsilon} + O\left(\frac{1}{\varepsilon^2}\right) = \frac{8}{\pi\varepsilon} + O\left(\frac{1}{\varepsilon^2}\right) = \frac{2.566}{\varepsilon} + O\left(\frac{1}{\varepsilon^2}\right). \tag{2.72}$$

The exact frequency under the limit  $\varepsilon \rightarrow \infty$  is

$$\omega = \frac{3.8929}{\varepsilon} + O\left(\frac{1}{\varepsilon^2}\right). \tag{2.73}$$

So the obtained solution is valid for both  $\varepsilon \rightarrow 0$  and  $\varepsilon \rightarrow \infty$ .

**2.5. Variational iteration method**

The variational iteration method<sup>70,74-77</sup> has been shown to solve effectively, easily, and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions. Most authors found that the shortcomings arising in Adomian method can be completely eliminated by the variational iteration method.<sup>6,7,17,96,97,103,111,112</sup>

To illustrate the basic idea of the variational iteration method, we consider the following general nonlinear system:

$$Lu + Nu = g(x), \tag{2.74}$$

where  $L$  is a linear operator, and  $N$  is a nonlinear operator.

According to the variational iteration method, we can construct the following iteration formulation:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \{Lu_n(s) + N\tilde{u}_n(s) - g(s)\} ds \tag{2.75}$$

where  $\lambda$  is called a general Lagrange multiplier, which can be identified optimally via the variational theory,  $\tilde{u}_n$  is considered as a restricted variation, i.e.  $\delta\tilde{u}_n = 0$ .

We consider Duffing equation as an example to illustrate its basic computational procedure.

$$\frac{d^2u}{dt^2} + u + \varepsilon u^3 = 0, \tag{2.76}$$

with initial conditions  $u(0) = A, u'(0) = 0$ .

Supposing that the angular frequency of the system (2.76) is  $\omega$ , we have the following linearized equation:

$$u'' + \omega^2 u = 0. \tag{2.77}$$

So we can rewrite Eq. (2.76) in the form

$$u'' + \omega^2 u + g(u) = 0, \tag{2.78}$$

where  $g(u) = (1 - \omega^2)u + \varepsilon u^3$ .

Applying the variational iteration method, we can construct the following functional:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda \{u_n''(\tau) + \omega^2 u_n(\tau) + \tilde{g}(u_n)\} d\tau, \tag{2.79}$$

where  $\tilde{g}$  is considered as a restricted variation, i.e.  $\delta\tilde{g} = 0$ .

Calculating variation with respect to  $u_n$ , and noting that  $\delta\tilde{g}(u_n) = 0$ , we have the following stationary conditions:

$$\begin{cases} \lambda''(\tau) + \omega^2 \lambda(\tau) = 0, \\ \lambda(\tau)|_{\tau=t} = 0, \\ 1 - \lambda'(\tau)|_{\tau=t} = 0. \end{cases} \tag{2.80}$$

The multiplier, therefore, can be identified as

$$\lambda = \frac{1}{\omega} \sin \omega(\tau - t). \tag{2.81}$$

Substituting the identified multiplier into Eq. (2.79) results in the following iteration formula:

$$u_{n+1}(t) = u_n(t) + \frac{1}{\omega} \int_0^t \sin \omega(\tau - t) \{u_n''(\tau) + u_n(\tau) + \varepsilon u_n^3(\tau)\} d\tau. \tag{2.82}$$

Assuming its initial approximate solution has the form:

$$u_0(t) = A \cos \omega t, \tag{2.83}$$

and substituting Eq. (2.83) into Eq. (2.76) leads to the following residual

$$R_0(t) = \left(1 - \omega^2 + \frac{3}{4}\varepsilon A^2\right) A \cos \omega t + \frac{1}{4}\varepsilon A^3 \cos 3\omega t. \tag{2.84}$$

By the formulation (2.82), we have

$$u_1(t) = A \cos \omega t + \frac{1}{\omega} \int_0^t R_0(\tau) \sin \omega(\tau - t) d\tau. \tag{2.85}$$

In order to ensure that no secular terms appear in  $u_1$ , resonance must be avoided. To do so, the coefficient of  $\cos \omega t$  in Eq. (2.84) requires to be zero, i.e.

$$\omega = \sqrt{1 + \frac{3}{4}\varepsilon A^2}. \tag{2.86}$$

So, from Eq. (2.85), we have the following first-order approximate solution:

$$\begin{aligned} u_1(t) &= A \cos \omega t + \frac{\varepsilon A^3}{4\omega} \int_0^t \cos 3\omega\tau \sin \omega(\tau - t) d\tau \\ &= A \cos \omega t + \frac{\varepsilon A^3}{32\omega^2} (\cos 3\omega t - \cos \omega t). \end{aligned} \tag{2.87}$$

It is obvious that for small  $\varepsilon$ , i.e.  $0 < \varepsilon \ll 1$ , it follows that

$$\omega = 1 + \frac{3}{8}\varepsilon A^2. \tag{2.88}$$

Consequently, in this limit, the present method gives exactly the same results as the standard Lindstedt–Poincaré method.<sup>98,100</sup> To illustrate the remarkable accuracy of the obtained result, we compare the approximate period

$$T = \frac{2\pi}{\sqrt{1 + 3\varepsilon A^2/4}} \tag{2.89}$$

with the exact one

$$T_{\text{ex}} = \frac{4}{\sqrt{1 + \varepsilon A^2}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k \sin^2 x}}, \tag{2.90}$$

where  $k = 0.5\varepsilon A^2/(1 + \varepsilon A^2)$ .

What is rather surprising about the remarkable range of validity of Eq. (2.89) is that the actual asymptotic period as  $\varepsilon \rightarrow \infty$  is also of high accuracy.

$$\lim_{\varepsilon \rightarrow \infty} \frac{T_{\text{ex}}}{T} = \frac{2\sqrt{3/4}}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - 0.5 \sin^2 x}} = 0.9294.$$

Therefore, for any value of  $\varepsilon > 0$ , it can be easily proved that the maximal relative error of the period Eq. (2.89) is less than 7.6%, i.e.  $|T - T_{\text{ex}}|/T_{\text{ex}} < 7.6\%$ .

Due to the very high accuracy of the first-order approximate solution, we always stop before the second iteration step. That does not mean that we cannot obtain higher order approximations.

### 3. Parameter-Expanding Methods

We consider the following nonlinear oscillator

$$mu'' + \omega_0^2 u + \varepsilon f(u) = 0. \quad (3.1)$$

Various perturbation methods have been applied frequently to analyze Eq. (3.1). The perturbation methods are, however, limited to the case of small  $\varepsilon$  and  $m\omega_0^2 > 0$ , that is, the associated linear oscillator must be statically stable in order that linear and nonlinear response be qualitatively similar.

Classical methods which are applied to weakly nonlinear systems include the Lindstedt–Poincare, Krylov–Bogoliubov–Mitropolski averaging, and multiple time scale methods. These are described by many textbooks by Bellman,<sup>15</sup> Nayfeh,<sup>98</sup> and Nayfeh and Mook,<sup>100</sup> among others. These methods are characterized by expansions of the dependent variables in power series in a small parameter, resulting in a collection of linear differential equations which can be solved successively. The solution to the associated linear problem thus provides a starting point for the generation of the “small perturbation” due to the nonlinearity, with various devices employed to annul secularities.

Classical techniques which have been used for both weakly and strongly nonlinear systems include the equivalent linearization method,<sup>105</sup> the harmonic balance method.<sup>79</sup> A feature of these methods is that the form of solution is specified in advance. The methods work well provided that the filter hypothesis is satisfied, that is, higher harmonics output by the nonlinearities are substantially attenuated by the linear part of the system.

#### 3.1. Modifications of Lindstedt–Poincare method

A number of variants of the classical perturbation methods have been proposed to analyze conservative oscillators similar to Eq. (3.1). The Shohat expansion<sup>108</sup> for the van der Pol equation is of remarkable accuracy not only for small parameter  $\varepsilon$ , but also for large  $\varepsilon$ . But recently Mickens<sup>95</sup> shows that it does not have a general validity as a “perturbation” scheme that can be extended to large values of  $\varepsilon$  for Duffing equation, as was originally thought to be. Burton<sup>23,24</sup> defined a new parameter  $\alpha = \alpha(\varepsilon)$  in such a way that asymptotic solutions in power series in  $\alpha$  converge more quickly than do the standard perturbation expansions in power series in  $\varepsilon$ .

$$\alpha = \frac{\varepsilon A^2}{4 + 3\varepsilon A^2}. \quad (3.2)$$

It is obvious that  $\alpha < 1/3$  for all  $\varepsilon A^2$ , so the expansion in power series in  $\alpha$  is quickly convergent regardless of the magnitude of  $\varepsilon A^2$ .

The Lindstedt–Poincare method<sup>98,100</sup> gives uniformly valid asymptotic expansions for the periodic solutions of weakly nonlinear oscillations, in general, the technique does not work in case of strongly nonlinear terms.

For the sake of comparison, the standard Lindstedt–Poincare method is briefly recapitulated first. The basic idea of the standard Lindstedt–Poincare method is to

introduce a new variable

$$\tau = \omega(\varepsilon)t, \tag{3.3}$$

where  $\omega$  is the frequency of the system. Equation (3.1) then becomes

$$\omega^2 mu'' + \omega_0^2 u + \varepsilon f(u) = 0, \tag{3.4}$$

where prime denotes differentiation with respect to  $\tau$ . The frequency is also expanded in powers of  $\varepsilon$ , that is

$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots. \tag{3.5}$$

Various modifications of Lindstedt–Poincare method appeared in open literature. It is of interest to note that a much more accurate relation for frequency  $\omega$  can be found by expanding  $\omega^2$ , rather than  $\omega$ , in a power series in  $\varepsilon$ .

$$\omega^2 = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots. \tag{3.6}$$

Cheung *et al.* introduced a new parameter

$$\alpha = \frac{\varepsilon\omega_1}{\omega_0^2 + \varepsilon\omega_1}. \tag{3.7}$$

and expand the solution and  $\omega^2$  in power series in  $\alpha$ .

All those modifications are limited to conservation system, instead of linear transformation (3.3), a nonlinear time transformation can be successfully annul such shortcoming.

We can re-write the transformation of standard Lindstedt–Poincare method in the form<sup>67</sup>

$$\tau = t + f(\varepsilon, t), \quad f(0, t) = 0, \tag{3.8}$$

where  $f(\varepsilon, t)$  is an unknown function, not a functional. We can also see from the following examples that the identification of the unknown function  $f$  is much easier than that of unknown functional  $F$  in Dai’s method.<sup>31,32</sup>

From the relation (3.8), we have

$$\frac{\partial^2 u}{\partial t^2} = \left(1 + \frac{\partial f}{\partial t}\right)^2 \frac{\partial^2 u}{\partial \tau^2} = G \frac{\partial^2 u}{\partial \tau^2}. \tag{3.9}$$

where

$$G(\varepsilon, t) = \left(1 + \frac{\partial f}{\partial t}\right)^2. \tag{3.10}$$

Applying the Taylor series, we have

$$G(\varepsilon, t) = G_0 + \varepsilon G_1 + \dots, \tag{3.11}$$

where  $G_i$  ( $i = 1, 2, 3, \dots$ ) can be identified in view of no secular terms. After identification of  $G_i$  ( $i = 1, 2, 3, \dots$ ), from the Eq. (3.10), the unknown function  $f$  can be solved. The transformation Eq. (3.8) can be further simplified as

$$\tau = f(t, \varepsilon), \quad f(t, 0) = t. \tag{3.12}$$

So we have

$$\frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial f}{\partial t}\right)^2 \frac{\partial^2 u}{\partial \tau^2}. \tag{3.13}$$

We expand  $(\partial f/\partial t)^2$  in a series of  $\varepsilon$ :

$$\left(\frac{\partial f}{\partial t}\right)^2 = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots. \tag{3.14}$$

Now we consider the van der Pol equation.

$$u'' + u - \varepsilon(1 - u^2)u' = 0. \tag{3.15}$$

By the transformation  $\tau = f(t, \varepsilon)$ , Eq. (3.15) becomes

$$\left(\frac{\partial f}{\partial t}\right)^2 \frac{\partial^2 u}{\partial \tau^2} + u = \varepsilon(1 - u^2) \frac{\partial f}{\partial t} \frac{\partial u}{\partial \tau}. \tag{3.16}$$

Introducing a new variable  $\omega$ , which is defined as

$$\left(\frac{\partial f}{\partial t}\right)^2 = \frac{1}{\omega^2}. \tag{3.17}$$

So Eq. (3.16) reduces to

$$\frac{\partial^2 u}{\partial \tau^2} + \omega^2 u = \varepsilon \omega(1 - u^2) \frac{\partial u}{\partial \tau}. \tag{3.18}$$

By simple operation, we have

$$\left(\frac{\partial f}{\partial t}\right)^2 = \frac{1}{\omega^2} = \frac{1}{1 + \frac{1}{8}\varepsilon^2}. \tag{3.19}$$

Solving Eq. (3.19), subject to the condition  $f(t, 0) = t$ , gives

$$\tau = f = \frac{t}{\sqrt{1 + \frac{1}{8}\varepsilon^2}}. \tag{3.20}$$

Its period can be expressed as

$$T = 2\pi\sqrt{1 + \frac{1}{8}\varepsilon^2}. \tag{3.21}$$

It is obvious that when  $\varepsilon \ll 1$ , the results are same as those obtained by standard perturbation methods.<sup>98</sup>

However, for large values of  $\varepsilon$  ( $\varepsilon \gg 1$ ), we have the following approximation<sup>98</sup> for exact period

$$T_{\text{ex}} \approx 2\varepsilon \int_{2/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{dv}{v} - 3v dv\right) = 1.614\varepsilon, \quad (\varepsilon \gg 1). \tag{3.22}$$

From Eq. (3.20), it follows

$$T \approx \frac{\pi}{\sqrt{2}}\varepsilon = 2.22\varepsilon, \quad (\varepsilon \gg 1). \tag{3.23}$$

It is obvious that for large  $\varepsilon$ , the approximate period has the same feature as the exact one, and we have

$$\lim_{\varepsilon \rightarrow \infty} \frac{T_{\text{ex}}}{T} = \frac{1.614}{2.22} = 0.727.$$

So the maximal relative error for all  $0 < \varepsilon < \infty$  is less than 37.5%. We can readily obtain higher-order approximations by MATHEMATICA.

Now we return to Eq. (3.1). Most methods are limited to odd nonlinearities and require that  $\omega_0^2 > 0$  and  $m > 0$ . We must devise a method to solve Eq. (3.1) for the case when (1)  $m > 0$ ,  $\omega_0^2 \leq 0$ ; (2)  $\omega_0^2 \geq 0$ ,  $m \leq 0$ ; (3) even nonlinearities exist. Parameter-expanding methods including modified Lindstedt–Poincare method<sup>67</sup> and bookkeeping parameter method<sup>64</sup> can successfully deal with such special cases where classical methods fail. The methods need not have a time transformation like Lindstedt–Poincare method, the basic character of the method is to expand the solution and some parameters in the equation. If we search for periodic solution of Eq. (3.1), then we can assume that

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots, \tag{3.24}$$

$$\omega_0^2 = \omega^2 + p\omega_1 + p^2\omega_2 + \dots, \tag{3.25}$$

$$m = 1 + pm_1 + p^2m_2 + \dots. \tag{3.26}$$

Hereby  $\varepsilon$  can be a small parameter in the equation, or an artificial parameter or a bookkeeping parameter.

As an illustration, consider the motion of a ball-bearing oscillating in a glass tube that is bent into a curve such that the restoring force depends upon the cube of the displacement  $u$ . The governing equation, ignoring frictional losses, is<sup>8</sup>

$$\frac{d^2u}{dt^2} + \varepsilon u^3 = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{3.27}$$

The standard Lindstedt–Poincare method does not work for this example. Now we re-write Eq. (3.27) in the form

$$u'' + 0 \cdot u + \varepsilon u^3 = 0. \tag{3.28}$$

Supposing the solution and the constant zero in Eq. (3.28) can be expressed as

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots, \tag{3.29}$$

$$0 = \omega^2 + \varepsilon c_1 + \varepsilon^2 c_2 + \dots. \tag{3.30}$$

Substituting Eqs. (3.29) and (3.30) into Eq. (3.28), and processing as the standard perturbation method, we have

$$u_0'' + \omega^2 u_0 = 0, \quad u_0(0) = A, \quad u_0'(0) = 0, \tag{3.31}$$

$$u_1'' + \omega^2 u_1 + c_1 u_0 = 0, \quad u_1(0) = 0, \quad u_1'(0) = 0. \tag{3.32}$$

Solving Eq. (3.31), we have  $u_0 = A \cos \omega t$ . Substituting  $u_0$  into Eq. (3.32) results into

$$u_1'' + \omega_2 u_1 + A \left( c_1 + \frac{3}{4} A^2 \right) \cos \omega t + \frac{1}{4} A^3 \cos 3\omega t = 0. \tag{3.33}$$

Eliminating the secular term needs

$$c_1 = -\frac{3}{4} A^2. \tag{3.34}$$

If only the first-order approximate solution is searched for, then from Eq. (3.30), we have

$$0 = \omega^2 - \frac{3}{4} \varepsilon A^2, \tag{3.35}$$

which leads to the result

$$\omega = \frac{\sqrt{3}}{2} \varepsilon^{1/2} A. \tag{3.36}$$

Its period, therefore, can be written as

$$T = \frac{4\pi}{\sqrt{3}} \varepsilon^{-1/2} A^{-1} = 7.25 \varepsilon^{-1/2} A^{-1}. \tag{3.37}$$

Its exact period can be readily obtained, which reads

$$T_{\text{ex}} = 7.4164 \varepsilon^{-1/2} A^{-1}. \tag{3.38}$$

It is obvious that the maximal relative error is less than 2.2%, and the obtained approximate period is valid for all  $\varepsilon > 0$ .

Now consider another example:<sup>99</sup>

$$u'' + \frac{u}{1 + \varepsilon u^2} = 0. \tag{3.39}$$

with initial conditions  $u(0) = A$  and  $u'(0) = 0$ .

We rewrite Eq. (3.39) in the form

$$u'' + 1 \cdot u + \varepsilon u'' u^2 = 0. \tag{3.40}$$

Proceeding with the same manipulation as the previous example, we obtain the following angular frequency:

$$\omega = \frac{1}{\sqrt{1 + \frac{3}{4} \varepsilon A^2}}. \tag{3.41}$$

It is obvious that for small  $\varepsilon$ , it follows that

$$\omega = 1 - \frac{3}{8} \varepsilon A^2. \tag{3.42}$$

Consequently, in this limit, the present method gives exactly the same results as the standard perturbations.<sup>99</sup> To illustrate the remarkable accuracy of the obtained results, we compare the approximate period

$$T = 2\pi \sqrt{1 + \frac{3}{4} \varepsilon A^2} \tag{3.43}$$

with the exact one

$$T = 4\sqrt{\varepsilon} \int_0^A \frac{du}{\sqrt{\ln(1 + \varepsilon A^2) - \ln(1 + \varepsilon u^2)}} = 4\sqrt{\varepsilon} \int_0^A \frac{du}{\sqrt{\ln[(1 + \varepsilon A^2)/(1 + \varepsilon u^2)]}}. \tag{3.44}$$

In the case  $\varepsilon A^2 \rightarrow \infty$ , Eq. (3.44) reduces to

$$\begin{aligned} \lim_{\varepsilon A^2 \rightarrow \infty} T &= 4\sqrt{\varepsilon} \int_0^A \frac{du}{\sqrt{2(\ln A - \ln u)}} = 2\sqrt{2\varepsilon}A \int_0^1 \frac{du}{\sqrt{\ln(1/s)}} \\ &= 4\sqrt{2\varepsilon}A \int_0^\infty \exp(-x^2)dx = 2\sqrt{2\pi\varepsilon}A. \end{aligned} \tag{3.45}$$

It is obvious that the approximate period (3.43) has the same feature as the exact one for  $\varepsilon \gg 1$ . And in the case  $\varepsilon \rightarrow \infty$ , we have

$$\lim_{A \rightarrow \infty} \frac{T_{\text{ex}}}{T} = \frac{2\sqrt{2\pi\varepsilon}A}{\sqrt{3\varepsilon\pi}A} = 0.9213.$$

Therefore, for any values of  $\varepsilon$ , it can be easily proven that the maximal relative error is less than 8.54% on the whole solution domain ( $0 < \varepsilon < \infty$ ).

### 3.2. Bookkeeping parameter

In most cases the parameter,  $\varepsilon$ , might not be a small parameter, or there is no parameter in the equation, under such conditions, we can use a bookkeeping parameter. Consider the mathematical pendulum:

$$u'' + \sin u = 0. \tag{3.46}$$

We re-write Eq. (3.46) in the following form

$$u'' + 0 \cdot u + \sin u = 0. \tag{3.47}$$

Supposing the solution and the constant, zero, can be expressed in the forms:

$$u = u_0 + pu_1 + p^2u_2 + \dots, \tag{3.48}$$

$$0 = \omega^2 + pc_1 + p^2c_2 + \dots. \tag{3.49}$$

Here,  $p$  is a bookkeeping parameter,  $c_i$  can be identified in view of no secular terms in  $u_i$  ( $i = 1, 2, 3, \dots$ ). For example, the constant  $c_1$  in Eq. (3.49) can also be identified in view of no secular terms in  $u_1$ , that is

$$\int_0^T A \cos \omega t \{c_1 A \cos \omega t + \sin(A \cos \omega t)\} dt = 0, \tag{3.50}$$

where  $T = 2\pi/\omega$ .

From Eq. (3.50),  $c_1$  can be determined explicitly by Bessel function. The angular frequency can be determined as

$$\omega = \sqrt{\frac{2J_1(A)}{A}}, \tag{3.51}$$

where  $J_1$  is the first-order Bessel function of the first kind.

The parameter-expanding methods can also be applied to non-periodic solutions. Consider another simple example.

$$u' + u^2 = 1, \quad u(0) = 0, \tag{3.52}$$

which has an exact solution

$$u_{\text{ex}} = \frac{1 - e^{-2t}}{1 + e^{-2t}}. \tag{3.53}$$

No small parameter exists in the equation, so in order to carry out a straightforward expansion, Liu<sup>88</sup> introduced an artificial parameter  $\varepsilon$  embedding in Eq. (3.52) so that he obtained the following equation

$$u' + (u + 1)(\varepsilon u - 1) = 0, \quad \varepsilon = 1. \tag{3.54}$$

Supposing that the solution can be expanded in powers of  $\varepsilon$ , and processing in a traditional fashion of perturbation technique, Liu obtained the following result<sup>88</sup>

$$u = (1 - e^{-t}) + \varepsilon e^{-t}(e^{-t} + t - 1). \tag{3.55}$$

Let  $\varepsilon = 1$  in Eq. (3.55). We obtain the approximate solution of the original Eq. (3.52):

$$u(t) = (1 - e^{-t}) + e^{-t}(e^{-t} + t - 1). \tag{3.56}$$

The obtained approximation (3.56) is of relative high accuracy.

In order to illustrate the basic idea of the proposed perturbation method, we rewrite Eq. (3.52) in the form

$$u' + 0 \cdot u + 1 \cdot u^2 = 1, \quad u(0) = 0. \tag{3.57}$$

Supposing that the solution and the coefficients (0 and 1) can be expressed in the forms:<sup>64</sup>

$$u = u_0 + pu_1 + p^2u_2 + \dots, \tag{3.58}$$

$$0 = a + pa_1 + p^2a_2 + \dots, \tag{3.59}$$

$$1 = pb_1 + p^2b_2 + \dots. \tag{3.60}$$

Substituting Eqs. (3.58)–(3.60) into Eq. (3.57), collecting the same power of  $p$ , and equating each coefficient of  $p$  to zero, we obtain

$$u'_0 + au_0 = 1, \quad u_0(0) = 0, \tag{3.61}$$

$$u'_1 + au_1 + a_1u_0 + b_1u_0^2 = 0, \tag{3.62}$$

$$u'_2 + au_2 + a_1u_1 + a_2u_0 + b_2u_0^2 + 2b_1u_0u_1 = 0. \tag{3.63}$$

We can easily solve the above equations sequentially for  $u_i$  ( $i = 1, 2, 3, \dots$ ), the initial conditions for  $u_i$  ( $i = 1, 2, 3, \dots$ ) should satisfy the condition  $\sum_{i=1} u_i(0) = 0$ .

Solving Eq. (3.61), we have

$$u_0 = \frac{1}{a}(1 - e^{-at}). \tag{3.64}$$

Substituting  $u_0$  into Eq. (3.62) results in

$$u_1' + au_1 + \frac{a_1}{a}(1 - e^{-at}) + \frac{b_1}{a^2}(1 - e^{-at})^2 = 0. \tag{3.65}$$

or

$$u_1' + au_1 + \left(\frac{a_1}{a} + \frac{b_1}{a^2}\right) - \left(\frac{a_1}{a} + \frac{2b_1}{a^2}\right)e^{-at} + \frac{b_1}{a^2}e^{-2at} = 0. \tag{3.66}$$

The requirement of no term  $te^{-at}$  in  $u_1$  needs

$$\frac{a_1}{a} + \frac{2b_1}{a^2} = 0. \tag{3.67}$$

If we keep terms to  $O(p^2)$  only, from Eqs. (3.59) and (3.60), we have

$$a + a_1 = 0 \quad \text{and} \quad b_1 = 1. \tag{3.68}$$

From the Eqs. (3.67) and (3.68), we obtain

$$a = \sqrt{2}. \tag{3.69}$$

Solving Eq. (3.66) subject to the initial condition  $u_1(0) = 0$  gives

$$u_1 = -\left(-\frac{1}{a} + \frac{1}{a^3}\right)(1 - e^{-at}) + \frac{1}{a^3}(e^{-2at} - e^{-at}). \tag{3.70}$$

Setting  $p = 1$ , we have the first-order approximation:

$$u(t) = u_0(t) + u_1(t) = \left(\frac{2}{a} - \frac{1}{a^3}\right)(1 - e^{-at}) + \frac{1}{a^3}(e^{-2at} - e^{-at}), \tag{3.71}$$

where  $a = \sqrt{2}$ . It is easy to prove that the obtained result (3.71) is uniformly valid on the whole solution domain.

Consider the Thomas–Fermi equation<sup>41,56,82</sup>

$$u''(x) = x^{-1/2}u^{3/2}, \quad u(0) = 1, \quad u(\infty) = 0. \tag{3.72}$$

The energy (in atomic units) for a neutral atom of atomic number  $Z$  is

$$E = \frac{6}{7} \left(\frac{4}{3\pi}\right)^{2/3} Z^{7/3}u'(0).$$

So it is important to determine a highly accurate value for the initial slope of the potential  $u'(0)$ .

The equation is studied by  $\delta$ -method.<sup>11</sup> The basic idea of the  $\delta$ -method is to replace the right hand side of the Thomas–Fermi equation by one which contains the parameter  $\delta$ , i.e.

$$u''(x) = u^{1+\delta}x^{-\delta}. \tag{3.73}$$

The solution is assumed to be expanded in a power series in  $\delta$

$$u = u_0 + \delta u_1 + \delta^2 u_2 + \dots \tag{3.74}$$

This, in turn, produces a set of linear equations for  $u_n$ :

$$\begin{aligned} u_0'' - u_0 &= 0, \\ u_1'' - u_1 &= u_0 \ln(u_0/x) \\ u_2'' - u_2 &= u_1 + u_1 \ln(u_0/x) + \frac{1}{2}u_0 \ln^2(u_0/x) \end{aligned}$$

with associated boundary conditions  $u_0(0) = 1, u_0(\infty) = 0$  and  $u_n(0) = u_n(\infty) = 0$  for  $n > 1$ .

To solve  $u_n$  for  $n > 1$ , we need some unfamiliar functions, so it might meet some difficulties in promoting this method.

We rewrite the equation in the form<sup>a</sup>

$$u''(x) - 0 \cdot u = 1 \cdot x^{-1/2}u^{3/2}. \tag{3.75}$$

Supposing that the solution and the coefficients (0 and 1) can be expressed in the forms

$$u = u_0 + pu_1 + p^2u_2 + \dots, \tag{3.76}$$

$$0 = \beta^2 + pa_1 + p^2a_2 + \dots, \tag{3.77}$$

$$1 = pb_1 + p^2b_2 + \dots. \tag{3.78}$$

By simple operation, we have the following equations

$$u_0''(x) - \beta^2u_0 = 0, \tag{3.79}$$

$$u_1''(x) - \beta^2u_1 = b_1x^{-1/2}u_0^{3/2} + a_1u_0, \tag{3.80}$$

with associated boundary conditions  $u_0(0) = 1, u_0(\infty) = 0$  and  $u_n(0) = u_n(\infty) = 0$  for  $n > 1$ . The first-order approximate solution is<sup>64</sup>

$$u_0 = e^{-1.19078255x}. \tag{3.81}$$

The second-order approximate solution is approximately solved as<sup>64</sup>

$$u(x) = \left(1 - \frac{1}{3}x\right) e^{-1.19078255x}. \tag{3.82}$$

In 1981, Nayfeh<sup>98</sup> studied the free oscillation of a nonlinear oscillation with quadratic and cubic nonlinearities, and the governing equation reads

$$u'' + u + au^2 + bu^3 = 0, \tag{3.83}$$

where  $a$  and  $b$  are constants.

<sup>a</sup>We can also assume that  $u''(x) + 0 \cdot u = 1 \cdot x^{-1/2}u^{3/2}$ , then the constant  $\beta^2$  will be proven to be negative.

In order to carry out a straightforward expansion for small but finite amplitudes for Eq. (3.83), we need to introduce a small parameter because none appear explicitly in this equation. To this end, we seek an expansion in the form<sup>98</sup>

$$u = qu_1 + q^2u_2 + q^3u_3 + \dots, \tag{3.84}$$

where  $q$  is a small dimensionless parameter that is a measure of the amplitude of oscillation. It can be used as a bookkeeping device and set equal to unity if the amplitude is taken to be small. As illustrated in Ref. 98, the straightforward expansion breaks down due to secular terms. And Nayfeh used the method of renormalization to render the straightforward expansion uniform. Unfortunately, the obtained results are valid only for small amplitudes.

Hagedorn<sup>43</sup> also used an artificial parameter to deal with a mathematical pendulum. For Eq. (3.83), we can assume that<sup>64</sup>

$$u = u_0 + pu_1 + p^2u_2 + \dots, \tag{3.85}$$

$$\omega_0^2 = \omega^2 + pc_1 + p^2c_2 + \dots, \tag{3.86}$$

$$a = pa_1 + p^2a_2 + \dots, \tag{3.87}$$

$$b = p^2b_1 + p^3b_2 + \dots. \tag{3.88}$$

We only write down the first-order angular frequency, which reads

$$\omega = \sqrt{\frac{1 + \frac{3}{4}bA^2 + \sqrt{(1 + \frac{3}{4}bA^2)^2 - \frac{5}{3}a^2A^2}}{2}}. \tag{3.89}$$

We write down Nayfeh’s result<sup>98</sup> for comparison:

$$\omega = 1 + \left(\frac{3}{8}b - \frac{5}{12}a^2\right)A^2, \tag{3.90}$$

which is obtained under the assumption of small amplitude. Consequently, in this limit, the present method gives exactly the same results as Nayfeh’s.<sup>98</sup>

#### 4. Parametrized Perturbation Method

In Refs. 73 and 71, an expanding parameter is introduced by a linear transformation:

$$u = \varepsilon v + b, \tag{4.1}$$

where  $\varepsilon$  is the introduced perturbation parameter,  $b$  is a constant.

We reconsider Eq. (3.52). Substituting Eq. (4.1) into Eq. (3.52) results in

$$\begin{cases} v' + \varepsilon v^2 + 2bv + (b^2 - 1)/\varepsilon = 0, \\ v(0) = -b/\varepsilon. \end{cases} \tag{4.2a}$$

Setting  $b = 1$  for simplicity, so we have the following equation with an artificial parameter  $\varepsilon$ :

$$\begin{cases} v' + \varepsilon v^2 + 2v = 0, \\ v(0) = -1/\varepsilon. \end{cases} \tag{4.2b}$$

Assume that the solution of Eq. (4.2b) can be written in the form

$$v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \dots \tag{4.3}$$

Unlike the traditional perturbation methods, we keep

$$v_0(0) = v(0), \quad \text{and} \quad \sum_{i=1} v_i(0) = 0. \tag{4.4}$$

We write the first-order approximate solution of Eq. (4.2b)

$$v = v_0 + \varepsilon v_1 = -\frac{1}{\varepsilon}e^{-2t} + \frac{1}{2\varepsilon}(e^{-4t} - e^{-2t}). \tag{4.5}$$

In view of the transformation (4.1), we obtain the approximate solution of the original Eq. (3.52):

$$u = \varepsilon v + 1 = 1 - e^{-2t} + \frac{1}{2}(e^{-4t} - e^{-2t}). \tag{4.6}$$

It is interesting to note that in Eq. (4.6) the artificial parameter  $\varepsilon$  is eliminated completely. It is obvious that even its first-order approximate solution (4.6) is of remarkable accuracy.

We now consider the following simple example to take the notation

$$u' + u - u^2 = 0, \quad u(0) = \frac{1}{2} \tag{4.7}$$

where there exists no small parameter. In order to apply the perturbation techniques, we introduce a parameter  $\varepsilon$  by the transformation

$$u = \varepsilon v \tag{4.8}$$

so the original Eq. (4.7) becomes

$$v' + v - \varepsilon v^2 = 0, \quad v(0) = A \tag{4.9}$$

where  $\varepsilon A = 1/2$ .

Supposing that the solution of Eq. (4.9) can be expressed in the form

$$v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \dots \tag{4.10}$$

and processing in a traditional fashion of perturbation technique, we obtain the following equations

$$v'_0 + v_0 = 0, \quad v_0(0) = A \tag{4.11}$$

$$v'_1 + v_1 - v_0^2 = 0, \quad v_1(0) = 0. \tag{4.12}$$

Solving Eqs. (4.11) and (4.12), we obtain

$$v_0 = Ae^{-t} \tag{4.13}$$

$$v_1 = A^2(e^{-t} - e^{-2t}). \tag{4.14}$$

Using the fact  $A = 1/(2\varepsilon)$ , we obtain the first-order approximate solution of Eq. (4.9)

$$\begin{aligned} v &= v_0 + \varepsilon v_1 + O(\varepsilon^2) = Ae^{-t} + \varepsilon A^2(e^{-t} - e^{-2t}) + O(\varepsilon^2) \\ &= \frac{1}{2\varepsilon}e^{-t} + \frac{1}{4\varepsilon}(e^{-t} - e^{-2t}) + O(\varepsilon^2). \end{aligned} \tag{4.15}$$

In view of the transformation (4.8), we have the approximate solution of the original Eq. (4.7)

$$u = \varepsilon v = \frac{3}{4}e^{-t} - \frac{1}{4}e^{-2t} \tag{4.16}$$

which is independent upon the artificial parameter  $\varepsilon$ , as it ought to be. The exact solution of Eq. (4.7) reads

$$u_{\text{ex}} = \frac{e^{-t}}{1 + e^{-t}} = e^{-t}(1 - e^{-t} + e^{-2t} + \dots). \tag{4.17}$$

It is obvious that the obtained approximate solution (3.16) is of high accuracy.

**Example 1.** Now we consider the Duffing equation with 5th order nonlinearity, which reads

$$u'' + u + \alpha u^5 = 0 \quad u(0) = A, \quad u'(0) = 0 \tag{4.18}$$

where  $\alpha$  needs not be small in the present study, i.e.  $0 \leq \alpha < \infty$ .

We let

$$u = \varepsilon v \tag{4.19}$$

in Eq. (4.18) and obtain

$$v'' + 1 \cdot v + \alpha \varepsilon^4 v^5 = 0, \quad v(0) = A/\varepsilon, \quad v'(0) = 0. \tag{4.20}$$

Applying the parameter-expanding method (modified Lindstedt–Poincare method<sup>67</sup>) as illustrated in Sec. 3.1, we suppose that the solution of Eq. (4.20) and the constant 1, can be expressed in the forms<sup>b</sup>

$$v = v_0 + \varepsilon^4 v_1 + \varepsilon^8 v_2 + \dots \tag{4.21}$$

$$1 = \omega^2 + \varepsilon^4 \omega_1 + \varepsilon^8 \omega_2 + \dots. \tag{4.22}$$

<sup>b</sup>Note: If we assume that

$$v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \varepsilon_4^v \dots \quad \text{and} \quad 1 = \omega^2 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \varepsilon^3 \omega_3 + \varepsilon^4 \omega_4 \dots$$

then it is easy to find that  $v_1 = v_2 = v_3 = 0$  and  $\omega_1 = \omega_2 = \omega_3 = 0$ , so that the secular terms will not occur.

Substituting Eqs. (4.21) and (4.22) into Eq. (4.20) and equating coefficients of like powers of  $\varepsilon$  yields the following equations

$$v_0'' + \omega^2 v_0 = 0, \quad v_0(0) = A/\varepsilon, \quad v_0'(0) = 0 \tag{4.23}$$

$$v_1'' + \omega^2 v_1 + \omega_1 v_0 + \alpha v_0^5 = 0, \quad v_1(0) = 0, \quad v_1'(0) = 0. \tag{4.24}$$

Solving Eq. (4.23) results in

$$v_0 = \frac{A}{\varepsilon} \cos \omega t. \tag{4.25}$$

Equation (4.24), therefore, can be re-written down as

$$v_1'' + \omega^2 v_1 + \left( \frac{5\alpha A^4}{8\varepsilon^4} + \omega_1 \right) \frac{A}{\varepsilon} \cos \omega t + \frac{5\alpha A^5}{16\varepsilon^5} \cos 3\omega t + \frac{\alpha A^5}{16\varepsilon^5} \cos 5\omega t = 0. \tag{4.26}$$

Avoiding the presence of a secular term needs

$$\omega_1 = -\frac{5\alpha A^4}{8\varepsilon^4} \tag{4.27}$$

Solving Eq. (4.26), we obtain

$$u_1 = -\frac{\alpha A^5}{128\varepsilon^5\omega^2}(\cos \omega t - \cos 3\omega t) - \frac{\alpha A^5}{384\varepsilon^5\omega^2}(\cos \omega t - \cos 5\omega t). \tag{4.28}$$

If, for example, its first-order approximation is sufficient, then we have

$$u = \varepsilon v = \varepsilon(v_0 + \varepsilon^4 v_1) = A \cos \omega t - \frac{\alpha A^5}{128\omega^2}(\cos \omega t - \cos 3\omega t) - \frac{\alpha A^5}{384\omega^2}(\cos \omega t - \cos 5\omega t) \tag{4.29}$$

where the angular frequency can be written in the form

$$\omega = \sqrt{1 - \varepsilon^4 \omega_1} = \sqrt{1 + \frac{5}{8}\alpha A^4}, \tag{4.30}$$

which is valid for all  $\alpha > 0$ .

**Example 2.** Consider the motion of a particle on a rotating parabola, which is governed by the equation (Exercise 4.8 in Ref. 99)

$$(1 + 4q^2 u^2) \frac{d^2 u}{dt^2} + \alpha^2 u + 4q^2 u \left( \frac{d^2 u}{dt^2} \right)^2 = 0, \quad t \in \Omega \tag{4.31}$$

with initial conditions  $u(0) = A$ , and  $u'(0) = 0$ .

Herein  $\alpha$  and  $q$  are known constants, and need not to be small. We let  $u = \varepsilon v$  in Eq. (4.31) and obtain

$$(1 + 4\varepsilon^2 q^2 v^2) \frac{d^2 v}{dt^2} + \alpha^2 v + 4q^2 \varepsilon^2 v \left( \frac{dv}{dt} \right)^2 = 0, \quad v(0) = A/\varepsilon, \quad v'(0) = 0. \tag{4.32}$$

By parameter-expanding method(the modified Lindsted–Poincare method<sup>67</sup>), we assuming that  $\alpha^2$  and solution of the Eq. (4.32) can be written in the form

$$\alpha^2 = \Omega^2 + \varepsilon^2\omega_1 + \varepsilon^4\omega_2 + \dots \tag{4.33}$$

$$v = v_0 + \varepsilon^2v_1 + \varepsilon^4v_2 + \dots \tag{4.34}$$

Substituting Eqs. (4.33) and (4.34) into Eq. (4.32), and equating the coefficients of like powers of  $\varepsilon$  results in the following equations

$$v_0'' + \Omega^2v_0 = 0, \quad v_0(0) = A/\varepsilon, \quad v_0'(0) = 0 \tag{4.35}$$

$$v_1'' + \Omega^2v_1 + \omega_1v_0 + 4q^2(v_0^2v_0'' + v_0v_0'^2) = 0, \quad v_1(0) = 0, \quad v_1'(0) = 0. \tag{4.36}$$

Solving Eq. (4.35) yields

$$v_0(\tau) = \frac{A}{\varepsilon} \cos \Omega t. \tag{4.37}$$

Equation (4.36), therefore, can be re-written down as

$$v_1'' + \Omega^2v_1 + \frac{A\omega_1}{\varepsilon} \cos \Omega t + \frac{4q^2A^3\Omega^2}{\varepsilon^3}(-\cos^3 \Omega t + \cos \Omega t \sin^2 \Omega t) = 0. \tag{4.38}$$

Using trigonometric identities, Eq. (4.38) becomes

$$v_1'' + \Omega^2v_1 + \left(\frac{A\omega_1}{\varepsilon} - \frac{2q^2A^3\Omega^2}{\varepsilon^3}\right) \cos \Omega t - \frac{2q^2A^3\Omega^2}{\varepsilon^3} \cos 3\Omega t = 0. \tag{4.39}$$

Eliminating the secular term demands that

$$\omega_1 = \frac{2q^2A^2\Omega^2}{\varepsilon^2}. \tag{4.40}$$

Solving Eq. (4.39), we obtain

$$v_1 = \frac{q^2A^3\Omega^2}{4\varepsilon^3}(\cos \Omega t - \cos 3\Omega t). \tag{4.41}$$

If, for example, its first-order approximation is sufficient, then we have the first-order approximate solution of Eq. (4.32)

$$u = \varepsilon(v_0 + \varepsilon^2v_1) = A \cos \omega t + \frac{q^2A^3\Omega^2}{4}(\cos \omega t - \cos 3\omega t) \tag{4.42}$$

where the angular frequency  $\omega$  can be solved from Eqs. (4.33) and (4.40)

$$\alpha^2 = \Omega^2 + \varepsilon^2\omega_1 = \Omega^2 + 2q^2A^2\Omega^2 \tag{4.43}$$

which leads to

$$\Omega = \frac{\alpha}{\sqrt{1 + 2(qA)^2}}. \tag{4.44}$$

Its period can be approximately expressed as follows

$$T_{\text{app}} = \frac{2\pi}{\alpha} \sqrt{1 + 2q^2A^2}. \tag{4.45}$$

In case  $qA$  is sufficiently small, i.e.  $0 < qA \ll 1$ , it follows that

$$T_{\text{app}} = \frac{2\pi}{\alpha}(1 + q^2 A^2).$$

Consequently, in this limit, the present method gives exactly the same results as the standard Lindstedt–Poincaré method.<sup>98</sup> However, in our present study,  $qA$  need not be small, even in case  $qA \rightarrow \infty$ , the present results also have high accuracy

$$\lim_{|qA| \rightarrow \infty} \frac{T_{\text{ex}}}{T_{\text{app}}} = \lim_{|qA| \rightarrow \infty} \frac{\frac{2}{\pi} \int_0^{\pi/2} \sqrt{1 + 4q^2 A^2 \cos^2 t} dt}{\sqrt{1 + 2q^2 A^2}} = \frac{2\sqrt{2}}{\pi} = 0.900 \quad (4.46)$$

where  $T_{\text{ex}}$  is the exact period, which reads

$$T_{\text{ex}} = \frac{2}{\alpha\pi} \int_0^{\pi/2} \sqrt{1 + 4q^2 A^2 \cos^2 t} dt. \quad (4.47)$$

Therefore, for any values of  $qA$ , it can be easily proven that

$$0 \leq \frac{|T_{\text{ex}} - T_{\text{app}}|}{T_{\text{app}}} \leq 10\%.$$

**Example 3.** Consider the equation (Exercise 4.4 in Ref. 99)

$$(1 + u^2)u'' + u = 0, \quad u(0) = A, \quad -u'(0) = 0. \quad (4.48)$$

We let  $u = \varepsilon v$  in Eq. (4.48) and obtain

$$v'' + 1 \cdot v + \varepsilon^2 v^2 v'' = 0, \quad v(0) = A/\varepsilon, \quad v'(0) = 0. \quad (4.49)$$

Supposing that the solution of Eq. (4.49) and  $\omega^2$  can be expressed in the form

$$v = v_0 + \varepsilon^2 v_1 + \varepsilon^4 v_2 + \dots \quad (4.50)$$

$$1 = \omega^2 + \varepsilon^2 \omega_1 + \varepsilon^4 \omega_2 + \dots. \quad (4.51)$$

Substituting Eqs. (4.50) and (4.51) into Eq. (4.49) and equating coefficients of like powers of  $\varepsilon$  yields the following equations

$$v_0'' + \omega^2 v_0 = 0, \quad v_0(0) = A/\varepsilon, \quad v_0'(0) = 0 \quad (4.52)$$

$$v_1'' + \omega^2 v_1 + \omega_1 v_0 + v_0^2 v_0'' = 0, \quad v_1(0) = 0, \quad v_1'(0) = 0. \quad (4.53)$$

Solving Eq. (4.52) results in

$$v_0 = \frac{A}{\varepsilon} \cos \omega t. \quad (4.54)$$

Equation (4.53), therefore, can be re-written down as

$$v_1'' + \omega^2 v_1 + \frac{\omega_1 A}{\varepsilon} \cos \omega t - \frac{\omega^2 A^3}{\varepsilon^3} \cos^3 \omega t = 0 \quad (4.55)$$

or

$$v_1'' + \omega^2 v_1 + \left( \frac{\omega_1 A}{\varepsilon} - \frac{3\omega^2 A^3}{4\varepsilon^3} \right) \cos \omega t - \frac{\omega^2 A^3}{4\varepsilon^3} \cos 3\omega t = 0. \quad (4.56)$$

We let

$$\omega_1 = \frac{3\omega^2 A^2}{4\varepsilon^2} \tag{4.57}$$

in Eq. (4.56) so that the secular term can be eliminated. Solving Eq. (4.56) yields

$$v_1 = \frac{A^3}{32\varepsilon^3}(\cos \omega t - \cos 3\omega t) \tag{4.58}$$

Thus we obtain the first-order approximate solution of the original Eq. (4.48), which reads

$$u = \varepsilon(v_0 + \varepsilon^2 v_1) = A \cos \omega t + \frac{A^3}{32}(\cos \omega t - \cos 3\omega t). \tag{4.59}$$

Substituting Eq. (4.57) into Eq. (4.51) results in

$$1 = \omega^2 + \varepsilon^2 \omega_1 = \omega^2 + \frac{3\omega^2 A^2}{4} \tag{4.60}$$

which leads to

$$\omega = \frac{1}{\sqrt{1 + \frac{3}{4}A^2}}. \tag{4.61}$$

To compare, we write Nayfeh’s result,<sup>98</sup> which can be written in the form

$$u = \varepsilon^{1/2} a \cos \left[ \left( 1 - \frac{3}{8}\varepsilon a^2 \right) t + \theta \right] \tag{4.62}$$

where  $\varepsilon$  is a small dimensionless parameter that is a measure of the amplitude of oscillation, the constants  $a$  and  $\theta$  can be determined by the initial conditions. Imposing the initial conditions, Nayfeh’s result can be re-written as

$$u = A \cos \left( 1 - \frac{3}{8}A^2 \right) t. \tag{4.63}$$

It is also interesting to point out that Nayfeh’s result is only valid only for small amplitude  $A$ , while the present results (4.59) and (4.61) are valid not only for small amplitude  $A$ , but also for very large values of amplitude.

### 5. Homotopy Perturbation Method

The homotopy perturbation method<sup>44–48,50–52,62,72</sup> provides an alternative approach to introducing an expanding parameter.

To illustrate its basic ideas of the method, we consider the following nonlinear differential equation

$$A(u) - f(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega \tag{5.1}$$

with boundary conditions

$$B(u, \partial u / \partial n) = 0, \quad \mathbf{r} \in \Gamma \tag{5.2}$$

where  $A$  is a general differential operator,  $B$  is a boundary operator,  $f(\mathbf{r})$  is a known analytic function,  $\Gamma$  is the boundary of the domain  $\Omega$ .

The operator  $A$  can, generally speaking, be divided into two parts  $L$  and  $N$ , where  $L$  is linear, while  $N$  is nonlinear. Equation (5.1), therefore, can be rewritten as follows

$$L(u) + N(u) - f(\mathbf{r}) = 0. \quad (5.3)$$

By the homotopy technique, we construct a homotopy  $v(\mathbf{r}, p) : \Omega \times [0, 1] \rightarrow \mathcal{R}$  which satisfies

$$\mathcal{H}(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(\mathbf{r})] = 0, \quad (5.4a)$$

or

$$\mathcal{H}(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(\mathbf{r})] = 0, \quad (5.4b)$$

where  $p \in [0, 1]$  is an embedding parameter,  $u_0$  is an initial approximation of Eq. (5.1), which satisfies generally the boundary conditions.

Obviously, from Eq. (5.4a) or Eq. (5.4b) we have

$$\mathcal{H}(v, 0) = L(v) - L(u_0) = 0, \quad (5.5)$$

and

$$\mathcal{H}(v, 1) = A(v) - f(\mathbf{r}) = 0. \quad (5.6)$$

It is obvious that when  $p = 0$ , Eq. (5.4a) or Eq. (5.4b) becomes a linear equation; when  $p = 1$  it becomes the original nonlinear one. So the changing process of  $p$  from zero to unity is just that of  $L(v) - L(u_0) = 0$  to  $A(v) - f(\mathbf{r}) = 0$ . The embedding parameter  $p$  monotonically increases from zero to unit as the trivial problem  $L(v) - L(u_0) = 0$  is continuously deformed to the problem  $A(v) - f(\mathbf{r}) = 0$ . This is a basic idea of homotopy method which is to continuously deform a simple problem easy to solve into the difficult problem under study.

The basic assumption is that the solution of Eq. (5.4a) or Eq. (5.4b) can be written as a power series in  $p$ :

$$v = v_0 + pv_1 + p^2v_2 + \cdots. \quad (5.7)$$

Setting  $p = 1$  results in the approximate solution of Eq. (5.1):

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \cdots. \quad (5.8)$$

Recently, some rather extraordinary virtues of the homotopy perturbation method have been exploited. The method has eliminated limitations of the traditional perturbation methods. On the other hand it can take full advantage of the traditional perturbation techniques so there has been a considerable deal of research in applying homotopy technique for solving various strongly nonlinear equations,<sup>2-4,25,29,37,109,110,120</sup> furthermore the differential operator  $L$  does not need to be linear, see Ref. 29.

Liao<sup>84–87</sup> proposes a kind of approximate solution technique based on the homotopy technique. In Liao’s approach, the solution is expanded into a more generalized Taylor series with a free parameter  $\hbar$ , a careful choice of the parameter  $\hbar$ , as illustrated in Liao’s publications, leads to fast convergence.

Different from Liao’s method, the homotopy perturbation method uses the imbedding parameter  $p$  as a small parameter, only few iterations (2 or 3) are needed to search for the needed asymptotic solution, unlike Liao’s method where infinite series is needed, the suitable choice of the free parameter  $\hbar$  can guarantee the convergence of the obtained series. Comparison between homotopy perturbation method and Liao’s method is systematically discussed in Ref. 51.

**5.1. Periodic solution**

As an illustrative example, we consider a problem of some importance in plasma physics concerning an electron beam injected into a plasma tube where the magnetic field is cylindrical and increases toward the axis in inverse proportion to the radius. The beam is injected parallel to the axis, but the magnetic field bends the path toward the axis. The governing equation for the path  $u(x)$  of the electrons is<sup>8</sup>

$$u'' + u^{-1} = 0, \tag{5.9}$$

with initial conditions  $u(0) = A, u'(0) = 0$ .

Physical considerations show that this equation has a periodic solution. In order to look for the periodic solution, we suppose that the frequency of Eq. (5.9) is  $\omega$ , and assume that the nonlinear term in Eq. (2.9) can be approximated by a linear term:<sup>60</sup>

$$u^{-1} \sim \omega^2 u. \tag{5.10}$$

So we obtain a linearized equation for Eq. (5.9)

$$u'' + \omega^2 u = 0. \tag{5.11}$$

We re-write Eq. (5.9) in the form

$$u'' + \omega^2 u = \omega^2 u - u^{-1}. \tag{5.12}$$

We construct the following homotopy

$$u'' + \omega^2 u = p(\omega^2 u - u^{-1}). \tag{5.13}$$

The embedding parameter  $p$  monotonically increases from zero to unit as the trivial problem, so Eq. (5.11) is continuously deformed to the original problem, Eq. (5.9). So if we can construct an iteration formula for Eq. (5.13), the series of approximations comes along the solution path, by incrementing the imbedding parameter from zero to one; this continuously maps the initial solution into the solution of the original Eq. (5.9).

Supposing that the solution of Eq. (5.13) can be expressed in a series in  $p$ :

$$u = u_0 + pu_1 + p^2u_2 + \dots. \tag{5.14}$$

We, therefore, obtain the following linear equations for  $u_0$  and  $u_1$ :

$$u_0'' + \omega^2 u_0 = 0, \quad u_0(0) = A, \quad u_0'(0) = 0, \tag{5.15}$$

$$u_1'' + \omega^2 u_1 = \omega^2 u_0 - u_0^{-1}, \quad u_1(0) = 0, \quad u_1'(0) = 0. \tag{5.16}$$

The solution of Eq. (5.15) is  $u_0(t) = A \cos \omega t$ . Substituting  $u_0$  into Eq. (5.16) results in

$$u_1'' + \omega^2 u_1 = A \omega^2 \cos \omega t - \frac{1}{A \cos \omega t}. \tag{5.17}$$

A rule of avoiding secular term in  $u_1$  leads to the condition

$$\int_0^{\pi/(2\omega)} \cos \omega t \left[ A \omega^2 \cos \omega t - \frac{1}{A \cos \omega t} \right] dt = 0, \tag{5.18}$$

which leads to the result

$$\omega = \sqrt{2}/A. \tag{5.19}$$

Applying the variational iteration method as illustrated in Sec. 2.5, we can solve  $u_1$  from Eq. (5.17) under the initial conditions  $u_1(0) = 0$  and  $u_1'(0) = 0$ ,

$$u_1(t) = \frac{1}{\omega} \int_0^t \sin \omega(s-t) \left[ A \omega^2 \cos \omega s - \frac{1}{A \cos \omega s} \right] ds. \tag{5.20}$$

So we obtain the first-order approximate solution by setting  $p = 1$ , which reads

$$u = u_0 + u_1 = A \cos \omega t + \frac{1}{\omega} \int_0^t \sin \omega(s-t) \left[ A \omega^2 \cos \omega s - \frac{1}{A \cos \omega s} \right] ds. \tag{5.21}$$

The period of the system can be expressed in the form

$$T = \sqrt{2}\pi A = 4.442A. \tag{5.22}$$

The exact period can be easily computed,

$$T_{\text{ex}} = 2\sqrt{2} \int_0^A \frac{du}{\sqrt{\int_u^A 1/s ds}} = 2\sqrt{2} \int_0^A \frac{du}{\sqrt{\ln A - \ln u}}. \tag{5.23}$$

By transformation  $u = As$ , Eq. (5.23) leads to

$$T_{\text{ex}} = 2\sqrt{2}A \int_0^1 \frac{du}{\sqrt{\ln(1/s)}} = 2\sqrt{2}\pi A = 5.01325A. \tag{5.24}$$

There exist alternative approaches to the construction of the needed homotopy. We rewrite Eq. (5.9) in the form

$$u''u^2 + u = 0. \tag{5.25}$$

We construct the following homotopy:

$$0 \cdot u'' + 1 \cdot u + pu''u^2 = 0. \tag{5.26}$$

Please note that we recover the main term  $u''$  by multiplying zero. Assuming the solution, the coefficients of  $u$  and  $u''$ , 1 and 0, respectively, can be expanded, in the following form

$$u = u_0 + pu_1 + p^2u_2 + \dots \tag{5.27}$$

$$1 = \omega^2 + pa_1 + p^2a_2 + \dots \tag{5.28}$$

$$0 = 1 + pb_1 + p^2b_2 + \dots \tag{5.29}$$

Substituting Eqs. (5.27)–(5.29) into Eq. (5.26) results in

$$(1 + pb_1 + \dots)(u''_0 + pu''_1 + \dots) + (\omega^2 + pa_1 + \dots)(u_0 + pu_1 + \dots) + p(u''_0 + pu''_1 + \dots)(u_0 + pu_1 + \dots)^2 = 0. \tag{5.30}$$

Collecting terms of the same power of  $p$ , gives

$$u''_0 + \omega^2u_0 = 0, \tag{5.31}$$

$$u''_1 + \omega^2u_1 + b_1u''_0 + a_1u_0 + u''_0u_0^2 = 0. \tag{5.32}$$

The solution of Eq. (5.31) is  $u_0(x) = A \cos \omega x$ . Substituting  $u_0$  into Eq. (5.32), by simple manipulation, yields

$$u''_1 + \omega^2u_1 - \left(b_1\omega^2 - a_1 + \frac{3}{4}A^2\omega^2\right)A \cos \omega x - \frac{1}{4}A^3\omega^2 \cos 3\omega x = 0. \tag{5.33}$$

No secular term requires that

$$b_1\omega^2 - a_1 + \frac{3}{4}A^2\omega^2 = 0. \tag{5.34}$$

If the first-order approximate solution is enough, then from Eqs. (5.28) and (5.29), we have

$$1 = \omega^2 + a_1 \tag{5.35}$$

$$0 = 1 + b_1. \tag{5.36}$$

Solving Eqs. (5.34)–(5.36) simultaneously, we obtain

$$\omega = \frac{2}{\sqrt{3}}A^{-1} = 1.1547A^{-1}. \tag{5.37}$$

The accuracy 7.8% is remarkable good.

We can also rewrite Eq. (5.9) in the form

$$1 + u''u = 0. \tag{5.38}$$

Multiplying both sides of Eq. (5.38) by  $u''$ , we have

$$u'' + u''^2u = 0. \tag{5.39}$$

We construct a homotopy in the form

$$u'' + 0 \cdot u + pu''^2u = 0. \tag{5.40}$$

Similarly, we expand the solution and the coefficient of the middle term, 0, into the forms

$$u = u_0 + pu_1 + p^2u_2 + \dots \tag{5.41}$$

$$0 = \omega^2 + pa_1 + p^2a_2 + \dots \tag{5.42}$$

As a result, we obtain

$$u''_0 + \omega^2u_0 = 0, \tag{5.43}$$

$$u''_1 + \omega^2u_1 + a_1u_0 + u''^2_0u_0 = 0. \tag{5.44}$$

The solution of Eq. (5.43) is  $u_0(x) = A \cos \omega x$ . Substituting  $u_0$  into Eq. (5.44), by simple manipulation, yields

$$u''_1 + \omega^2u_1 + \left(a_1 + \frac{3}{4}\omega^4A^2\right) A \cos \omega x + \frac{1}{4}\omega^4A^3 \cos 3\omega x = 0. \tag{5.45}$$

No secular terms in  $u_1$  requires that

$$a_1 = -\frac{3}{4}\omega^4A^2. \tag{5.46}$$

If we stop at the first-order approximate solution, then Eq. (5.42) becomes

$$0 = \omega^2 - \frac{3}{4}\omega^4A^2, \tag{5.47}$$

from which we can obtain the same result as Eq. (5.37).

Now we consider Duffing equation

$$u'' + u + \varepsilon u^3 = 0, \quad \varepsilon > 0 \tag{5.48}$$

with initial conditions  $u(0) = A$ , and  $u'(0) = 0$ .

We construct the following homotopy:

$$u'' + \omega^2u + p(\varepsilon u^3 + (1 - \omega^2)u) = 0, \quad p \in [0, 1]. \tag{5.49}$$

When  $p = 0$ , Eq. (5.49) becomes the linearized equation,  $u'' + \omega^2u = 0$ , when  $p = 1$ , it turns out to be the original one. We assume that the periodic solution to Eq. (5.49) may be written as a power series in  $p$ :

$$u = u_0 + pu_1 + p^2u_2 + \dots \tag{5.50}$$

Substituting Eq. (5.50) into Eq. (5.49), collecting terms of the same power of  $p$ , gives

$$u''_0 + \omega^2u_0 = 0, \quad u_0(0) = A, \quad u'_0(0) = 0 \tag{5.51}$$

$$u''_1 + \omega^2u_1 + \varepsilon u^3_0 + (1 - \omega^2)u_0 = 0, \quad u_1(0) = 0, \quad u'_1(0) = 0. \tag{5.52}$$

Equation (5.51) can be solved easily, giving  $u_0 = A \cos \omega t$ . If  $u_0$  is substituted into Eq. (5.52), and the resulting equation is simplified, we obtain

$$u''_1 + \omega^2u_1 + \left(1 + \frac{3}{4}\varepsilon A^2 - \omega^2\right) A \cos \omega t + \frac{1}{4}\varepsilon A^3 \cos 3\omega = 0. \tag{5.53}$$

No secular terms in  $u_1$  requires that

$$\omega = \sqrt{1 + \frac{3}{4}\varepsilon A^2}. \tag{5.54}$$

The accuracy of the expression (5.54), is 7.6% when  $\varepsilon \rightarrow \infty$ .

Secular terms arise for higher-order approximate solutions so we always stop before the second iteration. If higher-order approximate solution is required, the parameter-expanding method (the modified Lindstedt–Poincare method<sup>67</sup>) can be applied. For example, we can construct the following homotopy

$$u'' + 1 \cdot u + p\varepsilon u^3 = 0, \tag{5.55}$$

and we expand the coefficient of the linear term into a series of  $p$ :

$$1 = \omega^2 + p\omega_1 + p^2\omega_2 + \dots \tag{5.56}$$

where  $\omega_i$  can be identified in view of absence of secular terms in  $u_i$ :

$$\omega_1 = -\frac{3}{4}\varepsilon A^2, \quad \omega_2 = -\frac{3}{128\omega^2}\varepsilon^2 A^4. \tag{5.57}$$

Substituting in Eq. (5.56) and setting  $p = 1$ , we have

$$1 = \omega^2 - \frac{3}{4}\varepsilon A^2 - \frac{3}{128\omega^2}\varepsilon^2 A^4, \tag{5.58}$$

or

$$\omega = \sqrt{\frac{1 + \frac{3}{4}\varepsilon A^2 + \sqrt{1 + \frac{3}{2}\varepsilon A^2 + \frac{5}{8}\varepsilon^2 A^4}}{2}}. \tag{5.59}$$

The accuracy of Eq. (5.59) reaches 5.9% when  $\varepsilon \rightarrow \infty$ .

We can also construct a homotopy in the form

$$(1 - p)(v'' + \omega^2 v - u''_0 - \omega^2 u_0) + p(v'' + v + \varepsilon v^3) = 0. \tag{5.60}$$

By simple operation, we have

$$v''_0 + \omega^2 v_0 - u''_0 - \omega^2 u_0 = 0, \tag{5.61}$$

$$v''_1 + \omega^2 v_1 - (v''_0 + \omega^2 v_0 - u''_0 - \omega^2 u_0)(v''_0 + v_0 + \varepsilon v_0^3) = 0. \tag{5.62}$$

We begin with

$$v_0 = u_0 = A \cos \omega t. \tag{5.63}$$

Substituting Eq. (5.63) into Eq. (5.62), we have

$$v''_1 + \omega^2 v_1 + a \left( -\omega^2 + 1 + \frac{3}{4}\varepsilon A^2 \right) \cos \omega t + \frac{1}{4}\varepsilon A^3 \cos 3\omega t = 0, \tag{5.64}$$

which is identical to Eq. (5.53).

**5.2. Bifurcation**

In this section, we consider the following Duffing oscillator in space<sup>86</sup>

$$u'' + \varepsilon(u - A^2u^3) = 0, \quad u(0) = u(\pi) = 0, \quad u(\pi/2) = 1, \quad \varepsilon \geq 0. \quad (5.65)$$

For any  $\varepsilon \geq 0$ , the above equation has the trivial solution  $u(t) = 0$ . The so-called bifurcation occurs when a nontrivial solution exists for some of values of  $\varepsilon$ .

The traditional perturbation method predicts that<sup>86</sup>

$$A = \pm 2\sqrt{\frac{\varepsilon - 1}{3}}, \quad \varepsilon \geq 1, \quad (5.66)$$

which is only valid for small values of  $\varepsilon$ .

According to the homotopy perturbation, we construct the following simple homotopy:

$$u'' + \varepsilon u - \varepsilon p A^2 u^3 = 0. \quad (5.67)$$

Assume that the solution can be expressed in the form

$$u = u_0 + pu_1 + p^2u_2 + \dots \quad (5.68)$$

In the parlance of the parameter-expanding method (the modified Lindstedt–Poincare method<sup>67</sup>), we expand the coefficient of  $u$  in the middle term of Eq. (5.67) in the form

$$\varepsilon = \omega^2 + p\omega_1 + p^2\omega_2 + \dots \quad (5.69)$$

Substituting Eqs. (5.68) and (5.69) into Eq. (5.67), and equating the terms with the identical powers of  $p$ , we can obtain a series of linear equations and write only the first two linear equations:

$$u_0'' + \omega^2 u_0 = 0, \quad u_0(0) = u_0(\pi) = 0, \quad u_0(\pi/2) = 1, \quad (5.70)$$

$$u_1'' + \omega^2 u_1 + \omega_1 u_0 - \varepsilon A^2 u_0^3 = 0, \quad u_1(0) = u_1(\pi) = 0, \quad u_1(\pi/2) = 0. \quad (5.71)$$

The general solution of Eq. (5.70) is

$$u_0(t) = a \cos \omega t + b \sin \omega t, \quad (5.72)$$

where  $a$  and  $b$  are constant. Adjoining to the boundary conditions:  $u_0(0) = u_0(\pi) = 0$ , and the additional condition  $u_0(\pi/2) = 1$ , we have  $a = 1$ ,  $b = 0$ , and  $\omega = 1$ . So the solution of  $u_0$  reads

$$u_0 = \sin t. \quad (5.73)$$

Substituting  $u_0$  into Eq. (5.71), we obtain a differential equation for  $u_1$ ,

$$u_1'' + u_1 + \omega_1 \sin t - \varepsilon A^2 \sin^3 t = 0. \quad (5.74)$$

Or

$$u_1'' + u_1 + \left(\omega_1 - \frac{3}{4}\varepsilon A^2\right) \sin t - \frac{1}{4}\varepsilon A^2 \sin 3t = 0. \quad (5.75)$$

No secular terms in  $u_1$  requires that

$$\omega_1 = \frac{3}{4}\varepsilon A^2. \tag{5.76}$$

The solution of Eq. (5.75), considering the boundary conditions  $u_1(0) = u_1(\pi) = 0$ , and the additional condition  $u_1(\pi/2) = 0$ , is

$$u_1 = -\frac{1}{32}\varepsilon A^2(\sin 3t + \sin t). \tag{5.77}$$

We, therefore, obtain the first-order approximate solution which reads

$$u = u_0 + u_1 = \sin t - \frac{1}{32}\varepsilon A^2(\sin 3t + \sin t), \tag{5.78}$$

and

$$\varepsilon = 1 + \frac{3}{4}\varepsilon A^2. \tag{5.79}$$

Since  $\varepsilon \geq 0$ , the above equation has no solution when  $\varepsilon < 1$ . However, when  $\varepsilon > 1$ , Eq. (5.79) has the solution

$$A = \pm \frac{2}{\sqrt{3}}\sqrt{1 - \frac{1}{\varepsilon}}. \tag{5.80}$$

Therefore, the so-called simple bifurcation occurs at  $\varepsilon = 1$ . Our result is the same with that obtained by Liao.<sup>86</sup>

### 6. Iteration Perturbation Method

In this section, we will illustrate a new perturbation technique coupling with the iteration method.<sup>68</sup> Consider the following nonlinear oscillation:

$$u'' + u + \varepsilon f(u, u') = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{6.1}$$

We rewrite Eq. (6.1) in the following form:

$$u'' + u + \varepsilon u \cdot g(u, u') = 0, \tag{6.2}$$

where  $g(u, u') = f/u$ .

We construct an iteration formula for the above equation:

$$u''_{n+1} + u_{n+1} + \varepsilon u_{n+1}g(u_n, u'_n) = 0, \tag{6.3}$$

where we denote by  $u_n$  the  $n$ th approximate solution. For nonlinear oscillation, Eq. (6.3) is of Mathieu type. We will use the perturbation method to find approximately  $u_{n+1}$ . The technique is called iteration perturbation method.<sup>68,71</sup>

Consider Eq. (3.27), and assume that initial approximate solution can be expressed as  $u_0 = A \cos \omega t$ , where  $\omega$  is the angular frequency of the oscillation, we rewrite Eq. (3.27) approximately as follows

$$\frac{d^2u}{dt^2} + \varepsilon A^2 u \cos^2 \omega t = 0, \tag{6.4a}$$

or

$$\frac{d^2u}{dt^2} + \frac{1}{2}\varepsilon A^2 u + \frac{1}{2}\varepsilon A^2 u \cos 2\omega t = 0, \tag{6.4b}$$

which is Mathieu type. Suppose that

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots, \tag{6.5}$$

$$\frac{1}{2}\varepsilon A^2 = \omega^2 + \varepsilon c_1 + \varepsilon^2 c_2 + \dots \tag{6.6}$$

and substituting Eqs. (6.5) and (6.6) into Eq. (6.4b), and equating the coefficients of the same power of  $\varepsilon$ , we have the following two differential equations for  $u_1$  and  $u_2$ :

$$u_1'' + \omega^2 u_1 + c_1 u_0 + \frac{1}{2}A^2 \cos 2\omega t u_0 = 0, \quad u_1(0) = 0, \quad u_1'(0) = 0, \tag{6.7}$$

$$u_2'' + \omega^2 u_2 + c_2 u_0 + c_1 u_1 + \frac{1}{2}A^2 \cos 2\omega t u_1 = 0, \quad u_2(0) = 0, \quad u_2'(0) = 0, \tag{6.8}$$

where  $u_0 = A \cos \omega t$ . Substituting  $u_0$  into Eq. (6.7), the differential equation for  $u_1$  becomes

$$u_1'' + \omega^2 u_1 + A \left( c_1 + \frac{1}{4}A^2 \right) \cos \omega t + \frac{1}{4}A^3 \cos 3\omega t = 0. \tag{6.9}$$

The requirement of no secular term requires that

$$c_1 = -\frac{1}{4}A^2. \tag{6.10}$$

Solving Eq. (6.9), subject to the initial conditions  $u_1(0) = 0$  and  $u_1'(0) = 0$ , yields the result:

$$u_1 = \frac{1}{32\omega^2}A^3(\cos 3\omega t - \cos \omega t). \tag{6.11}$$

We, therefore, obtain its first-order approximate solution, which reads

$$u = A \cos \omega t + \frac{\varepsilon A^3}{32\omega^2}(\cos 3\omega t - \cos \omega t) = A \cos \omega t + \frac{A}{24}(\cos 3\omega t - \cos \omega t) \tag{6.12}$$

where the angular frequency is determined from the relations (6.6) and (6.10), which reads

$$\omega = \frac{\sqrt{3}}{2}\varepsilon^{1/2}A. \tag{6.13}$$

To obtain the second-order approximate solution, we substitute  $u_0$  and  $u_1$  into Eq. (6.8), in the parlance of no secular, we find

$$c_2 = \frac{c_1 A^2}{32\omega^2} = -\frac{A^4}{128\omega^2}. \tag{6.14}$$

Substituting the determined  $c_1$  and  $c_2$  into Eq. (6.6), we have

$$\frac{1}{2}\varepsilon A^2 = \omega^2 - \frac{1}{4}\varepsilon A^2 - \frac{\varepsilon^2 A^4}{128\omega^2}, \tag{6.15}$$

which leads to the result:

$$\omega = 0.8174\varepsilon^{1/2}A. \tag{6.16}$$

However, the second-order period  $T = 7.6867\varepsilon^{-1/2}A^{-1}$  is not more accurate than the first-order one, because Eq. (6.4b) is an approximate one, the obtained result (6.16) is the approximate angular frequency of Eq. (6.4b), not that of the original one. So we need not search for higher-order approximations. To obtain approximate solutions with higher accuracy, we replace zeroth-order approximate solution by the following

$$u_0 = A \cos \omega t + \frac{A}{24}(\cos 3\omega t - \cos \omega t). \tag{6.17}$$

So the original Eq. (3.27) can be approximated by the linear equation

$$\frac{d^2u}{dt^2} + \varepsilon \left[ A \cos \omega t + \frac{A}{24}(\cos 3\omega t - \cos \omega t) \right]^2 u = 0. \tag{6.18}$$

By the same manipulation, we can identify the angular frequency  $\omega = 0.8475\varepsilon^{1/2}A$ , and obtain the approximate period  $T = 7.413\varepsilon^{-1/2}A^{-1}$ . The relative error is about 0.05%.

Now we re-consider Eq. (5.9) in a more general form:

$$\frac{d^2u}{dt^2} + \frac{c}{u} = 0. \tag{6.19}$$

We approximate the above equation by

$$\frac{d^2u}{dt^2} + \frac{c}{A^2 \cos^2 \omega t} u = 0, \tag{6.20}$$

or

$$u'' + \frac{2c}{A^2}u + u'' \cos 2\omega t = 0. \tag{6.21}$$

We introduce an artificial parameter in Eq. (6.21)

$$u'' + \frac{2c}{A^2}u + \varepsilon u'' \cos 2\omega t = 0. \tag{6.22}$$

And suppose that

$$\frac{2c}{A^2} = \omega^2 + \varepsilon c_1 + \varepsilon^2 c_2 + \dots, \tag{6.23}$$

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots, \tag{6.24}$$

we obtain the differential equation for  $u_1$ :

$$u_1'' + \omega^2 u_1 + c_1 u_0 + u_0'' \cos 2\omega t = 0, \tag{6.25}$$

or

$$\begin{aligned} u_1'' + \omega^2 u_1 &= -Ac_1 \cos \omega t + A\omega^2 \cos \omega t \cos 2\omega t \\ &= \left(-Ac_1 + \frac{A\omega^2}{2}\right) \cos \omega t + \frac{A\omega^2}{2} \cos 3\omega t. \end{aligned} \quad (6.26)$$

The requirement of no secular term needs

$$c_1 = \frac{\omega^2}{2}. \quad (6.27)$$

So we identify the angular frequency as follows

$$\omega = \sqrt{\frac{4c}{3A^2}}. \quad (6.28)$$

The approximate period can be written as

$$T = \frac{\sqrt{3}\pi A}{c^{1/2}} = \frac{5.44A}{c^{1/2}}. \quad (6.29)$$

Acton and Squire,<sup>8</sup> using the method of weighted residuals, obtained the following result:

$$T = \frac{16A}{3c^{1/2}} = \frac{5.33A}{c^{1/2}}. \quad (6.30)$$

Now we consider two-dimensional viscous laminar flow over an finite flat-plain governed by a nonlinear ordinary differential equation:<sup>61,86,87</sup>

$$f''' + \frac{1}{2}ff'' = 0, \quad x \in [0, +\infty], \quad (6.31)$$

with boundary conditions  $f(0) = f'(0) = 0$ , and  $f'(+\infty) = 1$ .

The prime in Eq. (6.31) denotes the derivatives with respect to  $x$  which is defined as

$$x = Y \sqrt{\frac{U}{\nu X}},$$

and  $f(x)$  is relative to the stream function  $\Psi$  by

$$f(x) = \frac{\Psi}{\sqrt{\nu U X}}.$$

Here,  $U$  is the velocity at infinity,  $\nu$  is the kinematic viscosity coefficient,  $X$  and  $Y$  are the two independent coordinates.

In order to obtain a perturbation solution of Blasius equation, we introduce an artificial parameter  $\varepsilon$  in Eq. (6.31)

$$f''' + \frac{1}{2}\varepsilon ff'' = 0. \quad (6.32)$$

Processing in a traditional way of perturbation technique, and supposing that  $f''(0) = \sigma$ , we obtain a solution of Eq. (6.32) in the form of a power series<sup>87</sup>

$$f(x) = \sum_{k=0}^{+\infty} \left(-\frac{1}{2}\right)^k \frac{A_k \sigma^{k+1}}{(3k+2)!} x^{3k+2}$$

where

$$A_k = \begin{cases} 1 & (k = 0 \text{ and } k = 1) \\ \sum_{r=0}^{k-1} \binom{3k-1}{3r} A_r A_{k-r-1} & (k \geq 2) \end{cases}$$

which is valid only for small  $x$ .

Now we begin with the initial approximate solution

$$f_0(x) = x - \frac{1}{b}(1 - e^{-bx}), \tag{6.33}$$

where  $b$  is an unknown constant.

Equation (6.31) can be approximated by the following equation:

$$f''' + \frac{1}{2} \left[ x - \frac{1}{b}(1 - e^{-bx}) \right] f'' = 0, \tag{6.34}$$

We rewrite Eq. (6.34) in the form

$$f''' + bf'' + \frac{1}{2} \left[ x - \frac{1}{b}(1 - e^{-bx}) - 2b \right] f'' = 0, \tag{6.35}$$

and embed an artificial parameter  $\varepsilon$  in Eq. (6.35):

$$f''' + bf'' + \frac{1}{2}\varepsilon \left[ x - \frac{1}{b}(1 - e^{-bx}) - 2b \right] f'' = 0. \tag{6.36}$$

Suppose that the solution of Eq. (6.36) can be expressed as

$$f = f_0 + \varepsilon f_1 + \dots, \tag{6.37}$$

we have the following linear equations:

$$f_0''' + bf_0'' = 0, \quad f_0(0) = f_0'(0) = 0, \quad \text{and} \quad f_0'(+\infty) = 1, \tag{6.38}$$

$$f_1''' + bf_1'' + \frac{1}{2} \left( x + \frac{1}{b}e^{-bx} - 2b - \frac{1}{b} \right) f_1'' = 0, \tag{6.39}$$

$$f_1(0) = f_1'(0) = 0, \quad \text{and} \quad f_1'(+\infty) = 0.$$

The solution of Eq. (6.38) is  $f_0(x) = x - (1 - e^{-bx})/b$ , substituting it into Eq. (6.39) results in

$$f_1''' + bf_1'' = -\frac{1}{2}(bx + e^{-bx} - 2b^2 - 1)e^{-bx}. \tag{6.40}$$

The constant  $b$  can be identified by the following expression:

$$\int_0^{+\infty} e^{-bx}(bx + e^{-bx} - 2b^2 - 1)e^{-bx}dx = 0. \tag{6.41}$$

So we have

$$\frac{1}{4b}\Gamma(2) + \frac{1}{3b} - \frac{1 + 2b^2}{2b} = 0, \tag{6.42}$$

which leads to the result:

$$b = 1/\sqrt{12} = 0.28867. \tag{6.43}$$

The exact solution of Eq. (6.40) is not required, the expression (6.41) requires no terms of  $xe^{-\beta x}$  ( $n = 1, 2, 3, \dots$ ) in  $f_1$ . So we can assume that the approximate solution of Eq. (6.40) can be expressed as

$$f_1(t) = Ae^{-bx} + \frac{1}{8}e^{-2bx} + B, \tag{6.44}$$

where the constants  $A$  and  $B$  can be identified from the initial conditions  $f_1(0) = f_1'(0) = 0$ , i.e.  $A = -1/4$  and  $B = 1/8$ .

By setting  $p = 1$ , we obtain the first-order approximate solution, i.e.

$$f''(x) = f_0''(x) + f_1''(x), \tag{6.45}$$

where  $f_1$  and  $f_2$  are defined by Eqs. (6.33) and (6.44) respectively.

A highly accurate numerical solution of Blasius equation has been provided by Howarth,<sup>78</sup> who gives the initial slop  $f_{ex}''(0) = 0.332057$ . Comparing the approximate initial slop:

$$f''(0) = f_0''(0) + f_1''(0) = 0.3095$$

we find that the relative error is 6.8%.

We can improve the accuracy of approximate solution to a high-order by the iteration technique. Now we use Eq. (6.45) as the initial approximate solution of Eq. (6.31):

$$\tilde{f}_0(x) = x - \frac{1}{b}(1 - e^{-bx}) - \frac{1}{4}e^{-bx} + \frac{1}{8}e^{-2bx} + \frac{1}{8}, \tag{6.46}$$

where  $b$  is undermined constant.

By parallel operation, the constant  $b$  can be identified by the following relation:

$$\int_0^{+\infty} e^{-bx} \left( bx + e^{-bx} - 2b^2 - 1 - \frac{1}{4}e^{-bx} + \frac{1}{8}e^{-2bx} + \frac{1}{8} \right) e^{-bx} dx = 0 \tag{6.47}$$

which leads to the result

$$b = 0.3062. \tag{6.48}$$

It is obvious that  $\tilde{f}_0''(0) = b + 0.25b^2 = 0.3296$  reach a very high accuracy, and the 0.73% accuracy is remarkable good.

## 7. Ancient Chinese Methods

China is one of the four countries with an ancient civilization. The ancient Chinese mathematicians had made greatest contributions to the development of human culture, however, essentially nothing of a primary nature has come down to West concerning ancient Chinese mathematics, little has been discussed on ancient Chinese mathematics in some of the most famous monographs on history of mathematics. So the present author feels strongly necessary to give a basic introduction to the great classics of ancient Chinese mathematics for our Western colleagues, who are unfamiliar with the Chinese language. This section concerns briefly some famous ancient Chinese algorithms, which might find wide applications in modern science. *Jiu Zhang Suan Shu*, Nine Chapters on the Art of Mathematics, comprises nine chapters and hence its title is the oldest and most influential work in the history of Chinese mathematics. It is a collection of 246 problems on agriculture, business procedure, engineering, surveying, solution of equations, and properties of right triangles. Rules of solution are given systematically, but there exists no proofs in Greek sense. As it was pointed by Dauben<sup>34</sup> that the *Nine Chapters* can be regarded as a Chinese counterpart to Euclid's *Elements*, which dominated Western mathematics in the same the *Nine Chapters* came to be regarded as the seminal work of ancient Chinese mathematics for nearly two millennia. Great classics, when revisited in the light of new developments, may reveal hidden pearls, as is the case with ancient Chinese method and He Chengtian's inequality, which will be discussed below.

### 7.1. Chinese method<sup>53</sup>

The Chinese algorithm is called the *Ying Buzu Shu* in Chinese, which was proposed in about 2nd century BC, known as the rule of double false position in West after 1202 AD. Chapter 7 of the *Nine Chapters* is the *Ying Buzu Shu* (lit. Method of Surplus and Deficiency; too much, too less), an ancient Chinese algorithm, which is the oldest method for approximating real roots of a nonlinear equation. To illustrate the basic idea of the method, we consider the 15th example in this chapter, which reads:

The weight of a jade with a cubic *chun* is 7 *liang*, while the weight of a stone with a cubic *chun* is 6 *liang*. There is a cubic boulder with a length of 3 *chun*, and a mass of 11 *jin* (1 *jin* = 16 *liang*). What is mass of the jade and the stone in the boulder?

**Answer:** the volume of the jade in the boulder is 14 cubic *chun*, the mass is 6 *jin* and 2 *liang*; the volume of the stone is 13 cubic *chun*, the mass is 4 *jin* and 14 *liang*.

**The solution procedure:** Assume that the boulder is made of jade only, then the mass is overestimated by 13 *liang*. If we assume that boulder is made of stone only, the mass is underestimated by 14 *liang*. According to the *Ying Buzu Shu*, the mass of jade can be determined.

In the language of modern mathematics, let  $x$  and  $y$  be the volumes of the jade and stone, respectively, then we have

$$x + y = 3^3 = 27$$

and

$$7x + 6y = 11 \times 16 = 176.$$

We write  $R(x, y) = 7x + 6y - 198$ , and let  $x_1 = 27, y_1 = 0$  (assume that the boulder is made of jade only), we have  $R_1(27, 0) = 13$ ; if let  $x_2 = 0, y_2 = 27$  (assume that boulder is made of stone only), then we obtain  $R_2(0, 27) = -14$ . According to the ancient Chinese method, we can calculate the value of  $x$  in the form

$$x = \frac{x_2 R_1 - x_1 R_2}{R_1 - R_2} = \frac{27 \times 14}{13 + 14} = 14.$$

The ancient Chinese method can also be powerfully applied to nonlinear problems. We consider the last example of Chapter 7. It reads:

There is a wall with a thickness of 5 *chi*. Two mice excavate a hole in the wall from both sides. The big mouse burrows 1 *chi* in first day, and doubles its headway everyday afterwards; while the small one burrows 1 *chi* in first day also, but halves its headway everyday afterwards. When the hole can be completed? What is the length for each mouse?

**Answer:** 2 and  $2/17$  days. The big mouse penetrates totally 3 and  $12/17$  *chi*, while the small one 1 and  $5/17$  *chi*.

**The solution procedure:** Assume that it needs two days, then 5 *chun* (1 *chun* = 0.1 *chi*) is left. If we assume that it needs three days, then 3 *chi* and 0.75 *chun* is redundant. According to the *Ying Buzu Shu*, the days needed for excavation can be determined.

$$\frac{30.75 \times 2 + 5 \times 3}{5 + 30.75} \approx 2 \frac{2}{17}.$$

Consider an algebraic equation,

$$f(x) = 0. \quad (7.1)$$

Let  $x_1$  and  $x_2$  be the approximate solutions of the equation, which lead to the remainders  $f(x_1)$  and  $f(x_2)$  respectively, the ancient Chinese algorithm leads to the result

$$x = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}. \quad (7.2)$$

Now we rewrite Eq. (7.2) in the form

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)} = x_1 - \frac{f(x_1)(x_1 - x_2)}{f(x_1) - f(x_2)}. \quad (7.3)$$

If we introduce the derivative  $f'(x_1)$  defined as

$$f'(x_1) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}, \quad (7.4)$$

then we have

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}, \tag{7.5}$$

which is the well-known Newton iteration formulation proposed by Newton (1642–1727). Chinese mathematicians had used the theory for more than one millennium. The Chinese algorithm seems to have some advantages over the well-known Newton iteration formula if the two points  $(x_1$  and  $x_2)$  locate two sides of the exact root, i.e.  $f(x_1) \cdot f(x_2) < 0$ . Consider an example

$$f(x) = \sin x. \tag{7.6}$$

We begin with  $x_1 = 0.1$  and  $x_2 = 4.7$ . It is obvious that  $f_1 = 0.1 > 0$  and  $f_2 = -1.0 < 0$ , so there is a root locating  $x \in (x_1, x_2)$ . By Eq. (7.2), we have

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)} = 1.43.$$

Calculating  $f(x_3)$  results in  $f_3 = 0.99 > 0$ , so the root locates at  $x \in (x_2, x_3)$ . Using Eq. (7.2), we have

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = 3.06.$$

We see the procedure converges very fast to the exact root, the accuracy reaches 2.5% by only two iterations even for very poor initial prediction.

Consider the Duffing equation

$$u'' + u + \varepsilon u^3 = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{7.7}$$

We use the trial functions  $u_1(t) = A \cos t$  and  $u_2 = A \cos \omega t$ , which are, respectively, the solutions of the following linear equations

$$u'' + \omega_1^2 u = 0, \quad \omega_1^2 = 1, \tag{7.8}$$

and

$$u'' + \omega_2^2 u = 0, \quad \omega_2^2 = \omega^2. \tag{7.9}$$

The residuals are

$$R_1(t) = \varepsilon A^3 \cos^3 t, \tag{7.10}$$

$$R_2(t) = A(1 - \omega^2) \cos \omega t + \varepsilon A^3 \cos^3 \omega t. \tag{7.11}$$

By the Chinese method, we have

$$\omega^2 = \frac{\omega_1^2 R_2(0) - \omega_2^2 R_1(0)}{R_2(0) - R_1(0)} = \frac{1 - \omega^2 + \varepsilon A^2 - \omega^2 \varepsilon A^2}{1 - \omega^2} = 1 + \varepsilon A^2, \tag{7.12}$$

i.e.,

$$\omega = \sqrt{1 + \varepsilon A^2}. \tag{7.13}$$

The approximate period reads

$$T = \frac{2\pi}{\sqrt{1 + \varepsilon A^2}}. \tag{7.14}$$

Table 1 illustrates the comparison between the approximate period with the exact one when  $\varepsilon A^2 < 1$ .

The obtained solution obtained by the Chinese method is valid for the whole solution domain, even in the case  $\varepsilon A^2 \rightarrow \infty$ , the accuracy reaches 7.6%.

### 7.2. He Chengtian’s method<sup>54</sup>

In a history book, it writes<sup>80</sup>

*He Chengtian uses 26/49 as the strong, and 9/17 as the weak. Among the strong and the weak, Chengtian tries to find a more accurate denominator of the fractional day of the Moon. Chengtian obtains 752 as the denominator by using 15 and 1 as weighting factors, respectively, for the strong and the weak. No other calendar can reach such a high accuracy after Chengtian, who uses heuristically the strong and weak weighting factors.*

The above statement is generally called *Tiao Ri Fa* (Lit. Method for Modification of Denominator). Here, we call it as He Chengtian’s interpolation or He Chengtian’s method. There exists many other famous interpolations in ancient Chinese mathematics.

In modern mathematical view, we illustrate the statement as follows:

According to the observation data, He Chengtian finds that

$$29\frac{26}{49} \text{ days} > 1 \text{ Moon} > 29\frac{9}{17} \text{ days}.$$

Using the weighting factors (15 and 1), He Chengtian obtains

$$\text{the fractional day} = \frac{26 \times 15 + 9 \times 1}{49 \times 15 + 17 \times 1} = \frac{399}{752},$$

so

$$1 \text{ Moon} = 29\frac{399}{752} \text{ days}.$$

The error is about 0.1 second, it is the most accurate in He Chengtian’s time. He Chengtian establishes *Yuan-Jia* Calendar, which is the most famous lunar calendar in ancient China. The calendar was lunar with intercalary months to keep in approximate synchronization with the seasons.

Table 1. Comparison of approximate period with the exact one.

$\varepsilon A^2$	0	0.042	0.087	0.136	0.190	0.25
$T_{\text{ex}}$	6.283	6.187	6.088	5.986	5.879	5.767
$T$ (Eq. 7.14)	6.283	6.155	6.026	5.895	5.760	5.620

He Chengtian uses 26/49 as the strong weighing factor because it is much more accurate than that of the weak weighing factor, 9/17. The observation shows that the accuracy of the former is about 0.005%, while the accuracy for the latter is about 0.22%. So the weighting factor for 26/49 is much larger than that for 9/17.

He Chengtian actually uses the following inequality:

If

$$\frac{a}{b} < x < \frac{d}{c}, \tag{7.15}$$

where  $a, b, c$  and  $d$  are real numbers, then

$$\frac{a}{b} < \frac{ma + nd}{mb + nc} < \frac{d}{c}, \tag{7.16}$$

and  $x$  is approximated by

$$x(m, n) = \frac{ma + nd}{mb + nc}, \tag{7.17}$$

where  $m$  and  $n$  are weighting factors. We call  $x(1, 1) = (a + d)/(b + c)$  the He Chengtian’s average of  $a/b$  and  $d/c$ .

Generally speaking  $m$  and  $n$  can be freely chosen, and the accuracy of the obtained result is better than  $a/b$  or  $d/c$ . For example, if we choose  $m = 100$  and  $n = 1$ , then the improved fractional day can be calculated as

$$\text{the fractional day} = \frac{26 \times 100 + 9 \times 1}{49 \times 100 + 17 \times 1} = \frac{2609}{4917}.$$

The accuracy is about 0.0043%, and is improved compared with that of 26/49. Equation (7.16) can be rewritten equivalently in the form

$$x = \frac{ma + nd}{mb + nc} = \frac{ka + d}{kb + c}, \tag{7.18}$$

where  $k = m/n$ .

It is obvious that

$$\lim_{k \rightarrow 0} x = \frac{d}{c}, \tag{7.19}$$

and

$$\lim_{k \rightarrow \infty} x = \frac{a}{b}. \tag{7.20}$$

The changing process of  $k$  from zero to infinite is just that of  $x$  from  $d/c$  to  $a/b$ . There must exist a certain value of  $k$ , while the corresponding value of  $x$  locates at its exact solution.

There exists no general rule of choosing the value of  $k$ . It should be larger than 1 when  $a/b$  is more accurate than  $a/b$ . The more accurates  $a/b$  is, the larger value of  $k$ . If the accuracy of  $a/b$  is lower than that of  $d/c$ , the value of  $k$  should be smaller than 1. The less accurate  $a/b$  is, the smaller value of  $k$ .

In He Chengtian’s time, ancient Chinese mathematicians knew that  $157/50 < \pi < 22/7$  ( $\pi = 3.14$  was suggested by Liu Hui (263), and  $\pi = 22/7$  was obtained by Zu Chongzhi and/or his son Zu Geng.)

Using the weighting factors 1 and  $9^c$ , we have

$$\pi = \frac{157 + 9 \times 22}{50 + 9 \times 7} = \frac{355}{113} = 3.1415929.$$

So we obtain

$$\frac{355}{113} < \pi < \frac{22}{7},$$

which was obtained by Zu Chongzhi (430–501).

**Example 1.** Consider the complete elliptic integral of the first kind

$$\int_0^{\pi/2} \sqrt{1 - k \sin^2 x} \, dx.$$

It is obvious that

$$\frac{\pi}{2} < \int_0^{\pi/2} \sqrt{1 - k \sin^2 x} \, dx < \frac{\pi}{2} \sqrt{1 - k}. \tag{7.21}$$

By He Chengtian’s method, we have

$$\begin{aligned} \int_0^{\pi/2} \sqrt{1 - k \sin^2 x} \, dx &= \frac{\pi}{2(p + q)} (p + q\sqrt{1 - k}) \\ &= \frac{\pi}{2(1 + \xi)} (1 + \xi\sqrt{1 - k}), \end{aligned} \tag{7.22}$$

where  $p$  and  $q$  are weighting factors,  $\xi = q/p$ .

In the case  $k = 0$ , we have  $\xi = 0$ , and when  $k = 1$ , it follows  $\xi = \frac{\pi - 2}{2}$ , so we can approximately determine  $\xi$  in the form:

$$\xi = \frac{\pi - 2}{2} k. \tag{7.23}$$

We, therefore, have

$$\int_0^{\pi/2} \sqrt{1 - k \sin^2 x} \, dx = \frac{\pi}{2 + (\pi - 2)k} \left( 1 + \frac{\pi - 2}{2} k \sqrt{1 - k} \right). \tag{7.24}$$

**Example 2.** Consider the equation

$$(1 + u^2)u'' + u = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{7.25}$$

We rewrite Eq. (7.25) in the form

$$u'' = -\frac{1}{1 + u^2} u. \tag{7.26}$$

<sup>c</sup>22/7 is more accurate than 157/50, so the value of weight factor for the former is bigger than that of the latter.

If we choose the trial-function in the form  $u = A \cos \omega t$ , where  $\omega$  is the frequency, then the maximal and minimal values of  $1/(1 + u^2)$  are, respectively, 1 and  $1/(1 + A^2)$ . So we immediately obtain

$$\frac{1}{1} < \omega^2 < \frac{1}{1 + A^2}. \tag{7.27}$$

According to He Chengtian’s interpolation, we set

$$\omega^2 = \frac{m + n}{m + n(1 + A^2)} = \frac{1}{1 + kA^2}, \tag{7.28}$$

where  $m$  and  $n$  are weighting factors,  $k = n/(m + n)$ .

So the frequency can be approximated as

$$\omega = \frac{1}{\sqrt{1 + kA^2}}. \tag{7.29}$$

To compare, we write Nayfeh’s result,<sup>98</sup> which can be written in the form

$$u = A \cos \left( 1 - \frac{3}{8}A^2 \right) t, \tag{7.30}$$

which is valid only for small amplitude  $A$ . To match Nayfeh’s result, we set  $k = 3/4$  in Eq. (7.29), yielding the result

$$\omega = \frac{1}{\sqrt{1 + \frac{3}{4}A^2}}. \tag{7.31}$$

The accuracy reaches 8.5% even when  $A \rightarrow \infty$ , which is remarkably good.

**Example 3.** Consider Duffing equation, which reads

$$u'' + u + \varepsilon u^3 = 0, \quad u(0) = A, \quad u'(0) = 0 \tag{7.32}$$

where  $\varepsilon$  needs not be small in the present study, i.e.  $0 \leq \varepsilon < \infty$ .

We rewrite Eq. (7.32) in the form

$$u'' = -(1 + \varepsilon u^2)u, \tag{7.33}$$

which has a similar form of  $u'' = -\omega^2 u$ . Assume that period solution of Eq. (7.32) can be written in the form

$$u = A \cos \omega t, \tag{7.34}$$

where  $\omega$  is the frequency.

Observe that the square of frequency,  $\omega^2$ , is never less than that in the solution

$$\varphi_1(t) = A \cos t \tag{7.35}$$

of the following oscillation

$$u'' = -(1 + \varepsilon u_{\min}^2)u = -u. \tag{7.36}$$

In addition,  $\omega^2$  never exceed the square of frequency of the solution

$$\varphi_2(t) = A \cos \sqrt{1 + \varepsilon A^2} t \tag{7.37}$$

of the following oscillation

$$u'' = -(1 + \varepsilon u_{\max}^2)u = -(1 + \varepsilon A^2)u. \tag{7.38}$$

Hence, it follows that

$$\sqrt{\frac{1}{1}} < \omega < \sqrt{\frac{1 + \varepsilon A^2}{1}}. \tag{7.39}$$

According to He Chengtian's interpolation, we have

$$\omega = \sqrt{\frac{m + n(1 + \varepsilon A^2)}{m + n}} = \sqrt{1 + k\varepsilon A^2} \tag{7.40}$$

where  $k = n/(m + n)$ .

In the case  $\varepsilon \ll 1$ , the perturbation technique gives

$$\omega_{\text{pert}} = 1 + \frac{3}{8}\varepsilon A^2. \tag{7.41}$$

Matching Eq. (7.40) with Eq. (7.41) when  $\varepsilon \ll 1$ , we obtain  $k = 3/4$ . So the frequency can is obtained as follows

$$\omega = \sqrt{1 + \frac{3}{4}\varepsilon A^2}. \tag{7.42}$$

**Example 4.** Consider the motion of a particle on a rotating parabola, which is governed by the equation

$$(1 + 4q^2u^2)u'' + \varepsilon^2u + 4q^2uu'^2 = 0, \tag{7.43}$$

with initial conditions  $u(0) = A$ , and  $u'(0) = 0$ .

We rewrite Eq. (7.43) in the form

$$u'' = -f(u)u, \tag{7.44}$$

where

$$f(u) = \frac{\varepsilon^2 + 4q^2u'^2}{1 + 4q^2u^2}. \tag{7.45}$$

We begin with the trial function  $u(t) = A \cos \omega t$ , then we have

$$f(u) = \frac{\varepsilon^2 + 4q^2\omega^2A^2 \sin^2 \omega t}{1 + 4q^2A^2 \cos^2 \omega t}. \tag{7.46}$$

The maximum and minimum of  $f(u)$  are

$$f_{\max}(u) = \frac{\varepsilon^2}{1 + 4q^2A^2}, \tag{7.47}$$

$$f_{\min}(u) = \varepsilon^2, \tag{7.48}$$

respectively. So we have the following inequality

$$\sqrt{\frac{\varepsilon^2}{1}} < \omega < \sqrt{\frac{\varepsilon^2}{1 + 4q^2A^2}}. \tag{7.49}$$

By He Chengtian’s method, we have

$$\omega = \sqrt{\frac{(m + n)\varepsilon^2}{m + n(1 + 4q^2A^2)}} = \sqrt{\frac{\varepsilon^2}{1 + 4kq^2A^2}}. \tag{7.50}$$

The perturbation solution of Eq. (7.43) when  $\varepsilon \ll 1$  reads

$$\omega_{\text{pert}} = \frac{\varepsilon}{1 + 2q^2A^2}. \tag{7.51}$$

Comparison of Eq. (7.50) and (7.51) leads to the result:  $k = 1/2$ , so we obtain

$$\omega = \frac{\varepsilon}{\sqrt{1 + 2(qA)^2}}. \tag{7.52}$$

The accuracy reaches 10% even when  $qA \rightarrow \infty$ .

### 8. Conclusions

We have reviewed a few new asymptotic techniques with numerous examples. All reviewed methods can be applied to various kinds of nonlinear problems, and the examples studied in the present review article can be used as paradigms for real-life physics problems. For the nonlinear oscillators, all the reviewed methods yield high accurate approximate periods, but the accuracy of the amplitudes cannot be ameliorated by iteration, the discussion of the problem is available in *Journal of Sound and Vibration*, Vol. 282, pages 1317–1320 (2005).

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