

COMPLEXITY AND BOUNDED RATIONALITY IN INDIVIDUAL DECISION PROBLEMS

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Abstract

This paper develops a model of endogenous bounded rationality due to search costs arising from the complexity of the decision problem. The decision-maker is not required to know the entire structure of the problem when making choices. She can think ahead, through costly search, to reveal more details of the problem. The costs of search are not assumed exogenously, but inferred from revealed preferences through her choices. Thus, bounded rationality and its extent emerge endogenously: as problems become simpler or as the benefits of deeper search become larger relative to its costs, the choices more closely resemble those of a rational agent. For a fixed problem, the costs of search will vary across agents. This variation explains why the disparity between observed choices and those prescribed under rationality differs across decision-makers. Under additional assumptions, calibration of search costs suggests predictions and testable implications of the model. Through three applications, this approach of endogenous complexity costs is shown to be consistent with violations of timing independence in temporal framing problems, dynamic inconsistency and diversification bias in sequential versus simultaneous choice problems, and with plausible but contrasting risk attitudes across small- and large-stakes gambles.

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1 Introduction

Most economic models assume rational decision-making. Rationality, at a minimum, encompasses the assumption that the agent understands the entire decision problem, including all possible plans of action and the consequences of each, has well-defined preferences over final outcomes, and chooses optimally among the plans according to these preferences. For all but the simplest problems, making choices that match the predictions arising from rationality requires extraordinary cognitive and computational abilities and resources. The predictions of a variety of models are systematically violated across a spectrum of experimental and actual decision-makers. Addressing these discrepancies requires theories of bounded rationality that can account for agents with limited ability to formulate or solve complex problems, or to process information, as first advocated by Simon [42] in his pioneering work.

Economists have incorporated bounded rationality into game theory (Abreu and Rubinstein [1], Piccione and Rubinstein [28], Osborne and Rubinstein [27], Rosenthal [33], Camerer *et al.* [3]), bargaining (Sabourian [36]), auctions (Crawford and Iriberri [8]), macroeconomics (Sargent [39]), contracting (Anderlini and Felli [2]), to name but a few areas, in models that take the cognitive limitations as fixed and exogenous. However, an important feature of bounded rationality in practice is that the exact nature of limitations in cognitive, computational, or information-processing abilities vary both across decision-makers and across decision problems. More importantly, this variation can depend systematically on the nature of the decision problem. This suggests that it is the interaction between the cognitive limitations of the decision-maker and the precise structure of the decision problem that determines the extent of bounded rationality, and hence the resulting choices. Therefore, our understanding of the effects of bounded rationality on decision-making remains incomplete until we can relate the heterogeneity of cognitive abilities to the heterogeneity of different agents and different decision problems. Ultimately, the level of bounded rationality should itself be determined within the model. This requires explaining why a given agent departs from rational decision-making in some problems but not in others, and why in a given decision problem some agents exhibit bounded rationality while others do not. Ideally, this variation should be connected to observable features, yielding concrete implications which could, in principle, be tested empirically.

This paper takes a step towards these desiderata for general sequential decision problems of finite length. The decision-maker is not required to know the entire structure of the problem when making choices. She can think ahead, through costly search, to reveal more details of the problem. The costs of search are not assumed exogenously, but inferred from revealed preferences through her choices. Thus, bounded rationality and its extent emerge endogenously: as problems become simpler or as the benefits of deeper search become larger relative to its costs, the choices more closely resemble those of a rational agent. In fact, for a given agent, if the problem is sufficiently simple or the benefits of search in the problem outweigh its costs, the agent's choices become

indistinguishable from those of a rational agent. For a fixed problem, the costs of search will vary across agents. This variation explains why the disparity between observed choices and those prescribed under rationality differs across decision-makers.

Under certain assumptions, calibration of search costs suggests predictions and testable implications for the model. Specifically, the costs of complexity can be calibrated from observed responses when a given decision problem is presented across a population of agents. In addition, under a particular specification for the cost function (see Section 3.3), searching deeper into the problem terminates as soon as the marginal benefits of extending search begin to outweigh its marginal costs. Then, when the decision problem under study allows the agent's foresight to be revealed from observed choices, the marginal complexity costs can be partially identified. They are bounded from above by the benefit that would be accrued from extending the foresight beyond its current level and from below by the marginal benefit that would be lost from shortening the foresight below its current level. By varying the marginal benefits of further search along the decision tree, these bounds can be sharpened until an estimated value for the corresponding marginal costs is identified.

For a given decision problem, an estimate of the average marginal costs of complexity across a population of decision-makers allows for predictions. We can predict how changes in the terminal payoffs affect how far ahead into the problem an individual thinks and, consequently, what her responses are. Moreover, and perhaps equally importantly, the model is falsifiable as it dictates that the length of foresight is weakly increasing in the benefits of further search. This implies also that, for a given decision problem, there is a critical level for the benefits of further search which would induce deeper search. For a given agent and under additional assumptions, this allows us to transport the search costs calibrated in one decision-making situation to others of similar complexity, generating predictions regarding the depth of search in these decision-making situations.

Through three applications, I show that this approach of endogenous complexity costs is consistent with violations of timing independence in temporal framing problems, dynamic inconsistency and diversification bias in sequential versus simultaneous choice problems, and with plausible but contrasting risk attitudes across small- and large-stakes gambles. Each case corresponds to a well-known paradox in decision-making. The three decision-making situations are seemingly unrelated and the paradoxes obtain for obviously psychologically different reasons. Accordingly, a variety of models and arguments have been suggested in the literature to explain them. Yet, the complexity costs approach provides a common framework for depicting the underlying limitations that might force an agent's choices to depart from those prescribed by rational decision-making. In each case, paradoxical behavior results when the complexity costs are large relative to the benefits of search, preventing a boundedly-rational agent from considering the problem in its entirety.

Turning to a more specific description, the model considers a sequential decision problem of finite length with uncertainty, given in its extensive form. The payoffs to terminal nodes are given

and, for simplicity, agents are presumed to have expected utility preferences over these payoffs.¹ At any decision node, agents may have limited foresight, modeled as a restriction on the forward depth of the decision-tree that is considered when selecting a choice at the current decision node. To abstract from other aspects of departures from rationality (such as errors in choice, misperceptions of payoffs, behavioral or psychological biases in preferences or information-processing), I assume that agents always correctly perceive the terminal payoffs that fall within their current foresight. Outside their search horizon, however, decision-makers may not know either the structure of the continuation problem, or their payoffs from different actions. They are assumed to know only the worst possible payoff beyond the horizon for each available course of action within the horizon, and to use this worst case scenario to evaluate their options. Extending the foresight may involve complexity costs, and departures from rational choices reveal the presence of such costs. Given the decision-maker's choices, these costs can be identified.

To illustrate, consider the decision problem depicted in Figure 1. As in all of the figures throughout the paper, decision nodes (i) are depicted by squares while chance nodes (n) are depicted with circles. The terminal nodes in this example are indexed $\{A, \dots, K\}$, and the values in brackets at the terminal nodes represent the payoffs to the decision maker at each terminal node. The decision maker must choose a plan of action at his initial node $i.1$, given a specification f of distributions at each chance node. For example, suppose f assigns probability $1/2$ to each branch at $n.3$ and $n.4$, while assigning probability 1 to Right at $n.1$ and to Down at $n.2$. In this case, with perfect foresight and unbounded rationality, the decision maker will choose middle at $i.1$, and up at each other decision node, yielding an expected payoff of $3/4$.

Suppose instead that the decision maker chooses up at $i.1$. Such a choice can be rationalized with limited foresight and bounded rationality as the outcome of the model introduced in this paper. In this model, the search process divides the decision tree into horizons, with a horizon of foresight t consisting of all chains of information sets starting at a given node with length at most t . For example, the horizon of foresight 1 at node $i.1$, denoted $(1; i.1)$, is given by the collection of nodes $\{n.1, n.2, n.3\}$, while the horizon of foresight 2 , denoted $(2; i.1)$, is $(1; i.1) \cup \{i.2, i.3\}$. In this example, a horizon of foresight $t = 4$ is sufficient to replicate the standard perfect foresight case. For the horizon $(1; i.1)$, the agent knows the payoffs at terminal nodes A, B and K , as they directly succeed nodes within the horizon. Thus the agent can correctly assess the payoffs to choosing either up or down at $i.1$. In contrast, to middle the agent assigns the worst possible payoff from terminal nodes in the continuation problem following this choice, that is, the minimum payoff -1 over the nodes C, D, \dots, H . A horizon with larger foresight, such as $(2; i.1)$, gives the agent more precise information about some of the consequences of his choices, but also could be more costly. The choice of up at $i.1$, for example, reveals that the complexity cost to the agent of extending his horizon from $t = 1$ to $t = 4$ is at least $3/4$. Alternatively, a decision maker who chooses middle at $i.1$ but up at $i.4$ with some probability less than 1 has lower costs for extending from $t = 1$ to $t = 2$

¹Alternative non-expected utility preferences can also easily be accommodated. See Section 3.2.

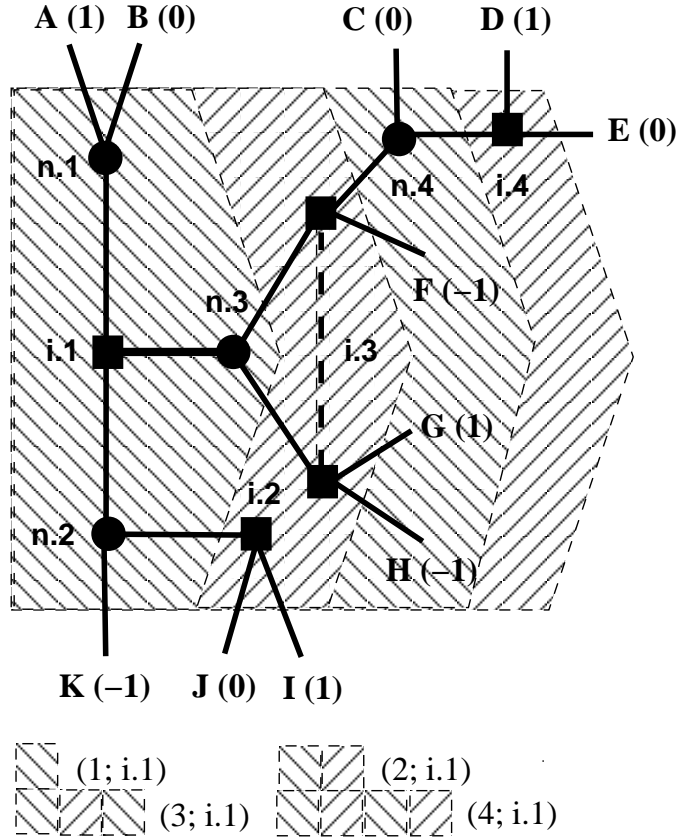


Figure 1: Behavior under Limited Foresight

at $i.1$, but still must have marginal costs at least $1/2$ from extending from $t = 2$ to $t = 4$.

This behavior under limited foresight amounts to a two-stage optimization. In the first stage, optimal plans against f are chosen for each length of foresight t . In the second, the agent chooses the optimal foresight. The process is, in principle, indistinguishable from rational decision-making. This can lead to quite different choices, however, if the cognitive limitations that necessitate search also render it costly. Departures from rational plans of action obtain when costs limit search to a horizon of inadequate foresight.

The paper fits into the large literature on bounded rationality in economics. In most of this work, however, aspects of cognitive limitations leading to bounded rationality are taken to be exogenously fixed. MacLeod [24] develops a search cost model of bounded rationality, drawing on work in computer and cognitive sciences, in which both payoffs and search costs are modeled by an exogenously given net value function and search proceeds by a heuristic algorithm (see also Eboli [11]). Models of limited foresight in repeated games, such as Rubinstein [35], and Jehiel [18], [19], take foresight to be exogenously given, implying that the complexity of the stage-game matters equally throughout the repeated game. When the set of contingencies that define the stage-

game is not a singleton, the number of contingencies within a horizon increases in a combinatorial manner with the number of periods. In contrast, the notion of complexity introduced in this paper incorporates both the number of periods and the number of contingencies within a given period. More importantly, the foresight of search is determined endogenously, which is crucial for explaining behavior in the applications I examine. In the work on repeated games played by finite automata, the number of states of a machine is often used to measure complexity (for example, see Neyman [26], Rubinstein [34], Abreu and Rubinstein [1], Piccione and Rubinstein [28], Kalai and Stanford [21], Sabourian [36]). Another strand of finite automata research focuses on the complexity of response rules (Chatterjee and Sabourian [4], Gale and Sabourian [13]). While this notion of complexity is more comprehensive than the one presented here, it is unclear how it can be applied beyond repeated games.

The paper is organized as follows. Section 2 presents the general model. Section 3 develops results and implications, including an endogenous stopping rule for search, and discusses related literature in more detail. Applications are presented in Section 4 and Section 5 concludes.

2 Modeling Bounded Rationality

A decision problem can be viewed as a game between the agent, who is strategic, and Nature. The essential elements are a set of observable events, a set of possible plans of action for the agent and Nature, and a payoff function for the agent. It can be represented with a directed graph, $\Gamma = \langle \mathcal{D}, \mathcal{A} \rangle$, consisting of a set \mathcal{D} of nodes and a set \mathcal{A} of arcs. The elements of \mathcal{A} correspond injectively to pairs of nodes $(d, d') : d \neq d', d, d' \in \mathcal{D}$.

If $(d, d') \in \mathcal{A}$, d' is an *immediate successor* of d . Let Ω_T denote the set of terminal nodes and $\mathcal{H} \subset \mathcal{D} \setminus \Omega_T$ be the collection of the sets of nodes at which the same player chooses from the same set of actions. \mathcal{H} forms a partition of $\mathcal{D} \setminus \Omega_T$ into information sets. The subset of the information sets where the agent acts will be denoted by H . Thus, $\mathcal{H} \setminus H$ is the set of Nature's chance nodes. Let also $D(h)$ denote the collection of nodes comprising the information set $h \in \mathcal{H}$. I will refer to h as an *immediate successor* of $d \in \mathcal{D}$, denoted by $d \triangleright h$, if some $d' \in D(h)$ is an immediate successor of d . Similarly, $h' \in \mathcal{H} : h' \neq h$ will be an *immediate successor* of h , denoted by $h \triangleright h'$, if there exists $d \in D(h)$ and $d' \in D(h')$ with $d \triangleright d'$.

2.1 Limited Foresight

The model of decision-making studied here is based on limited foresight. Given Nature's play, the agent will choose from her set of strategic alternatives to optimize with respect to some subset of the set of future contingencies. Specifically, she considers some part of the decision-tree that lies ahead of her current decision node consisting of chains of arc-connected information sets.

For a positive integer t , a *path from d to h' of length t* is a sequence h_1, h_2, \dots, h_t in \mathcal{H} such that $h_t = h'$, $d \triangleright h_1$, and $h_k \triangleright h_{k+1} : 1 \leq k \leq t - 1$. It will be denoted by $d \triangleright \{h_\tau\}_{\tau=1}^t$. The union of

all paths from all $d \in D(h)$ with length at most t is given by

$$(t; h) = \bigcup_{d \in D(h)} \bigcup_{k \in \{1, \dots, t\}} \left\{ d \triangleright \{h_\tau\}_{\tau=1}^k \right\}$$

This is the *horizon of foresight t starting at the information set h* . The union of all horizons starting at h is the *continuation problem* starting at h , denoted by Γ_h .² For finite trees, $\Gamma_h = (T_h; h)$ for some positive integer T_h and the horizons can be ordered by proper set-inclusion:

$$t' > t \iff (t; h) \subset (t'; h) \quad \forall t, t' \in \{1, \dots, T_h\}$$

with the index set $\mathbf{T}_h = \{1, \dots, T_h\}$ being an equivalent ordering.

Let $\Omega_T[h]$ be the set of terminal nodes that may occur conditional on play having reached some $d \in D(h)$. If $(t; h) \subset \Gamma_h$, then $(t; h)$ defines a partition of $\Omega_T[h]$ into the collection of terminal nodes that are immediate successors of information sets within the horizon and its complement. The collection is given by

$$\Omega_T[t; h] = \{y \in \Omega_T[h] : \exists h' \in (t; h) \wedge d' \in D(h') \wedge d' \triangleright y\}$$

A terminal node $y \in \Omega_T[t; h]$ will be called *observable* within $(t; h)$.

For $h \in \mathcal{H}$, $A(h)$ denotes the set of available actions at each of the nodes in $D(h)$. A pure strategy is a mapping $s : H \rightarrow \times_{h \in H} A(h)$ assigning to each of the agent's information sets an available action. Let S be the set of such mappings in the game. A mixed strategy for the agent will be denoted by $\sigma \in \Sigma = \Delta(S)$. A deterministic play by Nature is a mapping $q : \mathcal{H} \setminus H \rightarrow \times_{d \in \mathcal{H} \setminus H} A(d')$ assigning to each chance node in $\mathcal{H} \setminus H$ a move. Q is the set of such mappings in the game and $f \in \Delta(Q)$ denotes a mixed strategy by Nature.

Let $p(d|\sigma, f)$ be the probability that the node d is reached under (σ, f) , and $p(d'|\sigma, f, d)$ be the conditional probability that d' is reached under (σ, f) if play started at d . If d' cannot follow from d or $p(d|\sigma, f) = 0$, set $p(d'|\sigma, f, d) = 0$. The set of terminal nodes that may occur under (σ, f) , if play starts at somewhere in the information set h , is given by

$$\Omega_T[h|\sigma, f] = \{y \in \Omega_T[h] : \exists d \in D(h) \wedge p(d, y|\sigma, f) > 0\}$$

The subset

$$\Omega_T[t; h|\sigma, f] = \{y \in \Omega_T[t; h] : \exists d \in D(h) \wedge p(h, y|\sigma, f) > 0\}$$

is the set of terminal nodes that may occur under (σ, f) , if play starts somewhere in the information set h , and are observable within $(t; h)$.

²This is the standard definition of a continuation problem which is to be distinguished from a subgame: a subgame is a continuation problem but not vice versa. A node $d \in \mathcal{D}$ can be the root of a subgame Γ_d if and only if $\langle \text{either } h \cap \Gamma_d = \emptyset \text{ or } h \in \Gamma_d \rangle \forall h \in \mathcal{H}$. Any player who moves at d or at any information set of a subgame must know that d has occurred. This requires that only singleton information sets be starting points for subgames.

EXAMPLE 2.1: Consider again the decision-tree depicted in Figure 1. Starting at $i.1$, four classes of foresight can be defined. For $t = 1$, $(1; i.1) = \{d.1, d.2, d.3\}$. For $t = 2$, there are two chains of information sets: $i.1 \triangleright \{d.2, i.2\}$ and $i.1 \triangleright \{d.3, i.3\}$; the collection $\{d.1, d.2, d.3, i.2, i.3\}$ defines $(2; i.1)$. One chain of length $t = 3$ starts at $i.1$: $i.1 \triangleright \{d.3, i.3, d.4\}$. The horizon $(3; i.1)$ is given by $\{d.3, i.3, d.4\} \cup (2; i.1)$. Finally, $\{d.3, i.3, d.4, i.4\} \cup (3; i.1)$ defines $(4; i.1)$ (giving the entire tree). The terminal nodes are $\Omega_T [i.1] = \cup_{j=A}^K \{j\}$. But $\Omega_T [1; i.1] = \{A, B, K\}$, $\Omega_T [2; i.1] = \Omega_T [1; i.1] \cup \{F, G, H, I, J\}$, and $\Omega_T [3; i.1] = \Omega_T [2; i.1] \cup \{C\}$, $\Omega_T [4; i.1] = \Omega_T [i.1]$.

Let σ denote a mixed strategy for the agent prescribing positive probabilities for each of the available actions at $i.1$, $i.2$, and $i.4$, but assigning probability 1 to Down at $i.3$. Let f be a mixed strategy by Nature placing positive probability on every branch at $n.3$ and $n.4$, but assigning probability 1 to Down at $n.2$ and Right at $n.1$. We have $\Omega_T [2; i.1 | \sigma, f] = \{F, H, B, K\} = \Omega_T [i.1 | \sigma, f]$, and $\Omega_T [1; i.1 | \sigma, f] = \{B, K\}$. If s is the pure strategy in the support of σ that plays the middle branch at $i.1$, then $\Omega_T [1; i.1 | s, f] = \emptyset$, $\Omega_T [2; i.1 | s, f] = \{F, H\} = \Omega_T [i.1 | s, f]$.

Similarly, $\Omega_T [i.3] = \{C, D, E, F, G, H\} = \Omega_T [2; i.3]$ and $\Omega_T [1; i.3] = \{C, F, G, H\}$. Moreover, $\Omega_T [1; i.3 | \sigma, f] = \{F, H\} = \Omega_T [i.3 | \sigma, f]$.

2.2 Behavior under Limited Foresight

In what follows, I present a model of sophisticated decision-making under limited foresight. Starting at some node in h , the agent considers some horizon $(t; h)$, instead of the entire continuation problem Γ_h . She is aware, though, that there might be currently unforeseen contingencies that may happen given $f \in \Delta(Q)$ and a choice $\sigma \in \Sigma$. Thus, she views any two strategies, resulting in the same play against f within $(t; h)$, as equivalent. By Kuhn's Theorem, this equivalence concept can be defined as follows:

Definition 1 For $h \in H$ and $(t; h) \subseteq \Gamma_h$, two strategies $\sigma, \sigma' \in \Sigma$ are $(t; h)$ -**equivalent**, denoted $\sigma \underset{(t; h)}{\simeq} \sigma'$, if both prescribe the same probability distribution over $D(h') \forall h' \in H \cap (t; h)$. That is,

$$\sigma \underset{(t; h)}{\simeq} \sigma' \quad \text{iff} \quad \sigma|_{(t; h)} = \sigma'|_{(t; h)}$$

For $\tilde{\sigma} \in \Sigma$, $\tilde{\sigma}|_{(t; h)}$ denotes the unique behavior strategy defined when $\tilde{\sigma}$ is restricted to $H \cap (t; h)$.

Regarding outcomes beyond the truncated tree $(t; h)$, the set of terminal nodes that may follow from any $d \in D(h)$ under (s, f) but are not observable within $(t; h)$ is given by

$$\bar{\Omega}_T [t; h | s, f] = \bigcup_{\substack{s' \underset{(t; h)}{\simeq} s}} \Omega_T [h | s', f] \setminus \Omega_T [t; h | s, f]$$

Observe that, if $f(q)$ denotes the probability assigned by f to Nature's pure play $q \in Q$, this can be written also as follows

$$\bar{\Omega}_T [t; h | s, f] = \bigcup_{q \in Q: f(q) > 0} \bigcup_{\substack{s' \underset{(t; h)}{\simeq} s}} \bar{\Omega}_T [h | s', q]$$

I will assume that the agent knows the worst payoff in $\bigcup_{s' \underset{(t;h)}{\approx} s} \bar{\Omega}_T [h|s', q]$, for each q in the support of f , and that she evaluates the set $\bar{\Omega}_T [t; h|s, f]$ according to the expected worst payoff. This assumption suggests that the agent is averse to the uncertainty regarding her own choices beyond $(t; h)$ given that she is planning to play s against f within the horizon. This is similar in spirit to, but weaker than, assuming that she knows the entire set of possible but currently unforeseen terminal contingencies $\bar{\Omega}_T [t; h|s, f]$ and is ambiguity averse, in the sense of Gilboa and Schmeidler [13], using a maxmin criterion with respect to all possible distributions on $\bar{\Omega}_T [t; h|s, f]$. In Section 3, I discuss the robustness of my results with respect to these two assumptions.

Formally, let $r : \Omega_T \rightarrow \mathbb{R}_{++}$ be the agent's payoff function representing preferences over terminal outcomes.³ With a slight abuse of notation, for any set Y of terminal nodes of a decision-tree, let

$$r(Y) = \sum_{y \in Y} r(y) \quad \text{and} \quad r(\emptyset) = 0$$

Given a pure strategy profile $(s, q) \in S \times Q$, $r(\Omega_T [h|s, q])$ is the agent's total payoff on $\Omega_T [h|s, q]$. Preferences over lotteries on terminal consequences admit an expected utility representation and

$$u(h|\sigma, f) = \sum_{s \in S} \sigma(s) \sum_{q \in Q} f(q) r(\Omega_T [h|s, q])$$

is the agent's expected payoff under (σ, f) , given that play reached h . Let

$$BR(f) = \arg \max_{\sigma \in \Sigma} u(h|\sigma, f)$$

denote the set of best-responses from Σ under rational decision-making.

Under limited foresight, this notation must be enriched to distinguish the payoffs associated with terminal nodes that occur within the horizon $(t; h)$ from the payoffs corresponding to terminal nodes beyond the horizon. Within $(t; h)$, the payoff from (s, f) is given by

$$r(\Omega_T [t; h|s, f]) = \sum_{q \in Q} f(q) r(\Omega_T [t; h|s, q])$$

Beyond $(t; h)$, it is given by the worst expected payoff in $\bar{\Omega}_T [h|s, f]$. That is, the total payoff is given by the mapping $U(\cdot; h|\cdot) : \mathbf{T}_h \times S \times \Delta(Q) \rightarrow \mathbb{R}$ with

$$\begin{aligned} U(t; h|s, f) &= r(\Omega_T [t; h|s, f]) \\ &+ \min_{s' \in S: s' \underset{(t;h)}{\approx} s} \{u(h|s', f) - r(\Omega_T [t; h|s', f])\} \end{aligned} \quad (1)$$

The second term above corresponds to the worst expected outcome, given that s is played against f within $(t; h)$. It will be called the *continuation value* of the horizon $(t; h)$ under (s, f) . Observe,

³Although it comes at no loss of generality, the requirement of strictly positive terminal payoffs is needed only for Proposition 1 and for the specification that gives Stigler's endogenous stopping rule (see Section 3.3).

though, that $\Omega_T [t; h|s', f] \equiv \Omega_T [t; h|s, f]$, for any $f \in \Delta(Q)$ and any pair $s, s' \in S$ such that $s' \underset{(t;h)}{\simeq} s$.⁴ Thus, we may also write

$$U(t; h|s, f) = \min_{s' \in S: \substack{s' \underset{(t;h)}{\simeq} s}} u(h|s', f) \quad (2)$$

Now a binary relation $\underset{h}{\succsim}$ on \mathbf{T}_h can be defined by

$$t \underset{h}{\succsim} t' \text{ iff } \max_{\sigma \in \Sigma} \sum_{s \in S} \sigma(s) U(t; h|s, f) \geq \max_{\sigma \in \Sigma} \sum_{s \in S} \sigma(s) U(t'; h|s, f)$$

This is the agent's preference relation over horizons starting at h , given f . By construction, it admits the following utility representation

$$V(t; h|f) = \max_{\sigma \in \Sigma} \sum_{s \in S} \sigma(s) U(t; h|s, f)$$

The correspondence

$$BR(t; h|f) = \arg \max_{\sigma \in \Sigma} \sum_{s \in S} \sigma(s) U(t; h|s, f)$$

gives the optimal choices from Σ against f for the horizon $(t; h)$. The optimization problem

$$\max_{t \in \mathbf{T}_h} V(t; h|f) \quad (3)$$

determines optimal pairs $\{(t^*; h), BR(t^*; h|f)\}$ of horizon and strategies in $\Sigma \times \Gamma_h$.

2.3 Bounded Rationality

The behavior under limited foresight, given by (3), is equivalent to rational decision-making. That is, at any $h \in H$ and against any $f \in \Delta(Q)$, (3) results in the same set of best responses from Σ as that chosen by a rational decision-maker. This obtains by construction since $\underset{h}{\succsim}$ exhibits a weak preference for larger horizons.

Lemma 1 *The relation $\underset{h}{\succsim}$ exhibits a weak preference for larger horizons. Specifically,*

$$(M) \quad t > t' \implies t \underset{h}{\succsim} t'$$

Proof. For $h \in H$, take any $t, t' \in \mathbf{T}_h$ with $t > t'$. Since $(t'; h) \subset (t; h)$,

$$\left\{ s' \in S : s' \underset{(t;h)}{\simeq} s \right\} \subseteq \left\{ s' \in S : s' \underset{(t';h)}{\simeq} s \right\}$$

⁴This is an immediate implication of the preceding definition of equivalence for strategies. Notice that $\Omega_T [t; h|\cdot, f]$ is well-defined on the equivalence classes of $\underset{(t;h)}{\simeq}$.

Thus, for any $(s, f) \in S \times \Delta(Q)$,

$$\min_{s' \in S: s' \underset{(t'; h)}{\approx} s} u(h|s', f) \leq \min_{s' \in S: s' \underset{(t; h)}{\approx} s} u(h|s', f)$$

which by (2) implies

$$U(t'; h|s, f) \leq U(t; h|s, f)$$

The result follows immediately. ■

By (M), we have $\max_{t \in \mathbf{T}_h} V(t; h|f) = V(T_h; h|f)$. In other words,

$$\max_{t \in \mathbf{T}_h} V(t; h|f) = \max_{\sigma \in \Sigma} \sum_{s \in S} \sigma(s) U(T_h; h|s, f) = \max_{\sigma \in \Sigma} \sum_{s \in S} \sigma(s) u(h|s, f)$$

An agent with preferences $\underset{h}{\succsim}$ on \mathbf{T}_h will choose the maximal horizon and, thus, the same set of best responses from Σ , against $f \in \Delta(Q)$, as that given by the rational objective $u(h|\cdot, f)$. This implies that the behavior given by (3) is indistinguishable from rational decision-making and can be taken as the rationality benchmark. That is, departures from rational choices can be equivalently depicted as departures from this behavior under limited foresight.

This observation suggests that the behavior given by (3) can be used to compare boundedly-rational choices with rational decision-making. Such comparison requires, however, some structure on the boundedly-rational choices.

Definition 2 A choice rule $\mathcal{C}(S|h, f) \subseteq \Sigma$ against $f \in \Delta(Q)$ at h is **boundedly-rational** if there exists $t \in \mathbf{T}_h \setminus \{T_h\}$ such that $\mathcal{C}(S|h, f) = BR(t; h|f)$.

Notice that the chosen responses under a boundedly-rational rule would be optimal for the optimization problem in (3), albeit with some appropriate transformation of its objective. This transformation could represent a variety of psychological, behavioral and cognitive factors (or simply limitations in ability) that force departures from rationality. Yet, since preferences over lotteries on the set of terminal consequences admit an expected utility representation, it suffices to consider a strictly-increasing, linear transformation of $U(t; h|\cdot, f)$. Let $\tilde{U}(t; h|\cdot, f) : S \rightarrow \mathbb{R}$ be defined by⁵

$$\tilde{U}(t; h|s, f) = g(t; h|f) U(t; h|s, f)$$

for some function $g(\cdot; h|\cdot) : \mathbf{T}_h \times \Delta(Q) \rightarrow \mathbb{R}_{++}$. The optimization problem now becomes

$$\max_{t \in \mathbf{T}_h} \max_{\sigma \in \Sigma} \sum_{s \in S} \sigma(s) \tilde{U}(t; h|(s, f)) = \max_{t \in \mathbf{T}_h} g(t; h|f) V(t; h|f) \quad (4)$$

⁵The general formulation should be $\tilde{U}(t; h|s, f) = g(t; h|f) U(t; h|s, f) + g_0(t; h|f)$ for some function $g_0(\cdot; h|\cdot) : \mathbf{T}_h \times \Delta(Q) \rightarrow \mathbb{R}$. But given (t, h, f) , $g_0(t; h|f)$ does not affect the choice of responses from Σ . Thus, $g_0(\cdot; h|\cdot)$ can be taken to be the zero function on $\mathbf{T}_h \times \Delta(Q)$.

The optimal foresight in (4) will be denoted, henceforth, by $t_h(f)$. The set of maximizers from Σ will be called the g -best response set, and denoted by $BR_g(h|f)$. This formulation embeds rational decision-making, as a special case, for a constant function $g(\cdot; h|f)$.

Proposition 1 Fix $f \in \Delta(Q)$ and define $\mathbf{T}_h(f) = \{t \in \mathbf{T}_h : t \geq t_h(f)\}$.

(4) represents $\overset{h}{\succsim}$ on $\mathbf{T}_h(f)$ if and only if $g(\cdot; h|f)$ is constant on $\mathbf{T}_h(f)$.

Proof. The “if” part is trivial since, for a constant $g(\cdot; h|f)$, the objective in (4) is an affine transformation of $V(t; h|f)$. For the “only if”, let $t_h(f)$ be optimal against f and suppose that (4) represents $\overset{h}{\succsim}$ on $\mathbf{T}_h(f)$. By property (M),

$$g(t; h|f) V(t; h|f) \geq g(t_h(f); h|f) V(t_h(f); h|f) \quad \forall t \geq t_h(f)$$

and by optimality

$$g(t; h|f) V(t; h|f) \leq g(t_h(f); h|f) V(t_h(f); h|f) \quad \forall t \in \mathbf{T}_h$$

Hence,

$$g(t; h|f) V(t; h|f) = g(t_h(f); h|f) V(t_h(f); h|f) \quad \forall t \in \mathbf{T}_h(f)$$

But $t_h(f)$ must be optimal against f also for the $V(\cdot; h|f)$ representation of $\overset{h}{\succsim}$. Applying the monotonicity and optimality arguments once more we get

$$V(t; h|f) = V(t_h(f); h|f) \quad \forall t \in \mathbf{T}_h(f)$$

The result is now immediate. ■

This proposition establishes the function $g(\cdot; h|f)$ as an *indicator* of bounded rationality. That is, the agent is boundedly-rational against f at h if and only if $g(\cdot; h|f)$ is not constant on $\mathbf{T}_h(f)$. The agent is rational if and only if $g(\cdot; h|\cdot)$ is constant on $\mathbf{T}_h^0 \times \Delta(Q)$, where

$$\mathbf{T}_h^0 = \left\{ t \in \mathbf{T}_h : t \geq \min_{f \in \Delta(Q)} t_h(f) \right\}$$

EXAMPLE 2.3: Consider a standard, finite intertemporal decision problem. Let the agent’s payoff against $f \in \Delta(Q)$ be given by some time-separable function $r_t : S \times \Delta(Q) \rightarrow \mathbb{R}$, $t \in \{1, \dots, T\}$. We have

$$u(h|\sigma, f) = \sum_{s \in S} \sigma(s) \sum_{t=1}^T r_t(s, f)$$

In this case, (4) becomes

$$\sum_{s \in S} \sigma(s) \sum_{t=1}^T g(t; h|f) r_t(s, f)$$

For $g(t; h|f) = \delta^{t-1}$ and $g(t; h|f) = \beta \delta^{t-1} : \beta, \delta \in (0, 1)$, this is equivalent to exponential and hyperbolic discounting respectively. Time-preference can be represented as bounded rationality.

2.4 Complexity and its Costs

In general, different horizons $(t; h) \in \Gamma_h$ impose different degrees of informational limitation. Thus, the particular horizon the agent considers should matter for her response against f . Yet, this is not always the case because the preference for larger horizons is not necessarily strict: $\Delta_t V(\cdot; h|f) \geq 0 \forall t \in \mathbf{T}_h$, where $\Delta_t V(t; \cdot) = V(t+1; \cdot) - V(t; \cdot)$. Depending on the structure of the problem and Nature's play, at some $t \in \mathbf{T}_h$ the marginal benefit to a larger horizon may be zero. In such a case, extending the horizon from $(t; h)$ to $(t+1; h)$ does not affect the set of optimal responses against f . This corresponds to what is intuitively a not complex decision-making situation.

Definition 3 Take $h \in H$ and $t \in \mathbf{T}_h \setminus \{T_h\}$. $f \in \Delta(Q)$ is $(t; h)$ -**complex** if

$$\Delta_t V(t; h|f) > 0$$

f is h -**complex** if $\{t \in \mathbf{T}_h : \Delta_t V(t; h|f) > 0\} \neq \emptyset$. At h , the decision problem is **complex** if there exists an h -complex $f \in \Delta(Q)$.

EXAMPLE 2.4: For the decision problem of Figure 1, suppose that Nature may move in either direction at $n.3$ and $n.4$ with equal probability, left at $n.1$ with probability $p \in [0, 1]$, and deterministically down at $n.2$. At $i.1$, if the agent chooses the upper branch, she expects a payoff of p . If she chooses the lower branch, her payoff will be -1 . The middle branch corresponds to a continuation payoff of -1 for $t = 1$ (the worst case scenario is her moving Down at $i.3$), 0.5 for $t \in \{2, 3\}$ (the worst case scenario is now her moving Up at $i.3$ but Down at $i.4$), and a payoff of 0.75 for $t = 4$.

If $p \geq 0.75$, the expected value of every horizon is p . The assumed play by Nature is not complex at $i.1$. Instead, if $p < 0.75$, the expected value of the horizon is p for $t = 1$, $\max\{p, 0.5\}$ for $t \in \{2, 3\}$, and 0.75 for $t = 4$. The new play by Nature is complex at $i.1$.⁶ \square

Consider now the function $C(\cdot; h|f) : \mathbf{T}_h \rightarrow \mathbb{R}$, defined by

$$C(t; h|f) = [1 - g(t; h|f)] V(t; h|f) \tag{5}$$

This function acquires some interesting properties when the notion of complexity defined above is linked with the representation in (4). Since $t_h(f)$ denotes an optimal horizon in (4), we have

$$\begin{aligned} V(t_h(f); h|f) - C(t_h(f); h|f) &\geq V(t; h|f) - C(t; h|f) \implies \\ C(t; h|f) - C(t_h(f); h|f) &\geq V(t; h|f) - V(t_h(f); h|f) \\ &\geq 0 \quad \forall t \in \mathbf{T}_h(f) \quad \text{by (M)} \end{aligned}$$

If we normalize $g(t_h(f); h|f) = 1$, then $C(t_h(f); h|f) = 0$ and the inequality establishes that $C(t; h|f) \geq 0$, for any $t \in \mathbf{T}_h(f)$ and $f \in \Delta(Q)$. In other words, $C(\cdot; h|\cdot)$ is a cost function on $\mathbf{T}_h(f)$. It will be called, henceforth, the *complexity cost function*.

⁶This example illustrates also that extreme uncertainty-aversion does not prohibit search *per se*. Although the worst outcome remains unchanged throughout the game, expanding the foresight of search may still be beneficial if it uncovers paths leading to higher rewards or making the worst outcome less likely to occur.

Lemma 2 Take $h \in H$ and $t \in \mathbf{M}_h \setminus \{T_h\}$. If $f \in \Delta(Q)$ is not $(t; h)$ -complex, then it cannot entail any marginal complexity costs on the horizon $(t; h)$. That is,

$$\Delta_t V(t; h|f) = 0 \implies \Delta_t C(t; h|f) = 0$$

Proof. Fix $f \in \Delta(Q)$. By (M), at any $h \in H$ and for any $t, t' \in \mathbf{M}_h$ and $s \in S$

$$U(t; h|s, f) \leq U(t+1; h|s, f)$$

Let $\sigma^* \in BR(t; h|f)$ and suppose that there exists $s' \in \text{support}\{\sigma^*\}$ such that the above holds as a strict inequality for s' . We have

$$\begin{aligned} \max_{\sigma \in \Sigma} \sum_{s \in S} \sigma(s) U(t+1; h|s, f) &\geq \sum_{s \in S} \sigma^*(s) U(t+1; h|s, f) \\ &> \sum_{s \in S} \sigma^*(s) U(t; h|s, f) \\ &= \max_{\sigma \in \Sigma} \sum_{s \in S} \sigma(s) U(t; h|s, f) \implies \\ V(t+1; h|f) &> V(t; h|f) \end{aligned}$$

which contradicts $\Delta_t V(t; h|f) = 0$. Thus, it must be

$$U(t+1; h|s', f) = U(t; h|s', f) \quad \forall s \in S : \sigma^*(s) > 0$$

which implies, however,

$$\begin{aligned} \sum_{s \in S} \sigma^*(s) U(t+1; h|s, f) &= \sum_{s \in S} \sigma^*(s) U(t; h|s, f) \\ &= \max_{\sigma \in \Sigma} \sum_{s \in S} \sigma(s) U(t; h|s, f) \\ &= V(t; h|f) = V(t+1; h|f) \end{aligned}$$

In other words, we established that

$$\Delta_t V(t; h|f) = 0 \implies BR(t; h|f) \subseteq BR(t+1; h|f) \tag{6}$$

For given $t \in \mathbf{M}_h$, $BR(t; h|f)$ is, however, the best-response set also for the transformed problem in (4). Therefore,

$$\begin{aligned} g(t+1; h|f) V(t+1; h|f) &= g(t; h|f) V(t; h|f) \xrightarrow{\Delta_t V(t; h|f)=0} \\ g(t+1; h|f) &= g(t; h|f) \end{aligned}$$

The result follows immediately. ■

Lemma 2 ensures that the complexity cost function does not affect the behavior under limited foresight in decision-making situations that are not complex. It affects optimal choices under

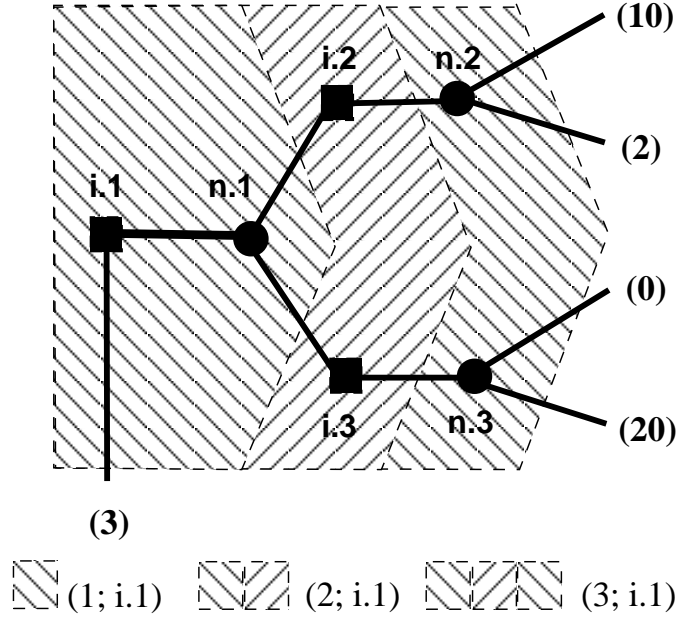


Figure 2: A Non-Complex Problem

limited foresight only when the agent would strictly prefer to expand her horizon if she were not constrained by her bounded rationality. In this sense, the function depicts the presence of (and only of) bounded rationality. To see why this feature is desirable, consider again Nature’s (non-strictly) mixed strategy described in the first part of Example 2.4. Facing this stochastic play, an agent who chooses the upper branch at $i.1$ may or may not be boundedly-rational. In this problem, the given play by Nature is simply not complex enough to reveal bounded rationality from observed choices. Therefore, any function depicting the presence of bounded rationality should not indicate this choice as boundedly-rational.

Another interesting property of the complexity cost function is that those contingencies beyond a given horizon which do not require strategic decisions by the agent do not add to the current costs. As the following example illustrates, calculating expectations with respect to Nature’s probability distributions is not costly. In this model, it cannot account for departures from rationality. Discrepancies from rational decision-making can result only from the the agent’s limited ability to handle the complexity of making decisions.

EXAMPLE 2.5: For the problem of Figure 2, consider the stochastic play f by Nature that places equal probability at the branches of every chance node. The agent has only one available move at $i.2$ and the problem is not complex at either of her two decision nodes. The value of having

foresight $t = 1$ is 8 (playing down at $i.1$ terminates the game within the reach of the horizon for a payoff of 3; playing right corresponds to an expected continuation payoff of 8). The best response and the value of the horizon remain unchanged for $t \geq 2$. That is, $\Delta_t V(t; i.k|f) = 0$ for every $t \in T_{i.k}$, $k \in \{1, 2\}$. Consequently, $\Delta_t C(\cdot; i.k|f) = 0$ on $T_{i.k}$. Notice that one may reduce the decision-tree so that it consists of only the information set $i.1$, without changing the complexity of the problem at hand. \square

In this paper, complexity is defined with respect to the structure of the decision problem. It does depend, therefore, upon the particular conjecture about Nature’s play. This dependence is reflected in the function $C(t; h|\cdot)$, allowing the model to exhibit an important endogenous form of bounded rationality. When playing a complex game against a strategic opponent, the perceived ability of the opponent seems to obviously matter when the agent decides how deeply into the game-tree to plan for. Even though Nature is not a strategic opponent, there is an analogue of this intuition in a decision-theoretic setting. Different conjectures about Nature’s play correspond to different sets of unforeseen contingencies that may occur beyond the current horizon. Hence, the difficulty of a given decision problem varies with the set of Nature’s stochastic plays considered by the agent. In this model, the conjectures about Nature’s play matter not only through the expected terminal payoffs but also directly through the complexity-cost function. This may lead a boundedly-rational agent to choose different horizons in response to different conjectures.

Allowing $C(t; h|\cdot)$ to vary on $\Delta(Q)$, for a given foresight $t \in \mathbf{T}_h$, also offers a built-in flexibility to differentiate between the length and breadth of a horizon, in terms of complexity. The cost of considering a horizon $(t; h)$ against a strictly mixed strategy $f \in \Delta(Q)$ by Nature may well be strictly higher than the corresponding cost against any pure strategy in the support of f . This could reflect the larger number of contingencies which may happen within the horizon against the mixture. While the foresight is t in either case, responding against f requires accounting for more contingencies.

Finally, the model allows $C(t; \cdot|f)$ to vary across the information sets at which decisions are to be made, against a given $f \in \Delta(Q)$. It can accommodate, therefore, decision-making situations where the complexity of the continuation problem varies at different stages of the problem.

3 Discussion and Related Literature

The preceding analysis viewed bounded rationality as resulting from costs introduced by the complexity of the decision problem, and showed that these implied costs obtain in such a way that decision-making under bounded rationality admits the following representation

$$\max_{t \in \mathbf{T}_h} V(t; h|f) - C(t; h|f) \tag{7}$$

Given $f \in \Delta(Q)$, this optimization problem identifies the optimal horizon $t_h(f)$. The resulting responses are given by the set $BR_g(h|f) = BR(t_h(f); h|f)$. In this section, I discuss some of the

strengths and limitations of this representation and how it relates to some of the literature.

3.1 Strictly Dominated Strategies Matter

This model assumes that the agent always knows the worst terminal contingency that may arise against $f \in \Delta(Q)$ when $s \in S$ is played within $(t; h)$, and uses the corresponding expected payoff to evaluate the set $\bar{\Omega}_T(t; h|s, f)$ of possible future terminal outcomes under (s, f) . An important implication is that the introduction of strictly dominated strategies can make the decision-maker strictly better or worse off.

For finite, non-repeated decision problems, any $s \in S$ will eventually reach a unique terminal node and, against $q \in Q$, a horizon will either include this node or not. That is, one of the two terms of the partition $\Omega_T[t; h|s, q] \cup \bar{\Omega}_T[t; h|s, q]$ will be the empty set, for any $(s, q) \in S \times Q$ and $(t; h) \in \mathbf{T}_h \times H$. There exists, therefore, $t_h(s, q) \in \mathbf{T}_h$ such that $g(t_h(s, q); h|s, q) = u(h|s, q)$. Suppose now that $\tilde{s} \in S$ does strictly worse than s against q conditional on play having reached h : $u(h|s, q) > u(h|\tilde{s}, q)$. By monotonicity, introducing \tilde{s} does not matter for the objective in (4) if $t \geq t_h(s, q)$. Taking $t \geq \max_{q \in Q} t_h(s, q)$, this is true for a strictly dominated \tilde{s} . For finitely-repeated problems, this argument suggests that \tilde{s} will again not matter if it is strictly dominated against q in the stage game and the foresight suffices to cover the stage game.

To illustrate, consider removing the option of moving Down at $i.3$ in Figure 1. Any strategy that uses this option with positive probability is strictly dominated by any other that prescribes moving Up with probability 1 at $i.3$. The deletion will not affect the valuation $V(t; i.1|f)$, for $t \geq 2$, because the dominance is revealed within the reach of the horizon. This is not true, however, for the horizon $(1; i.1|f)$. With respect to the play by Nature described in Example 2.4, we have $V(1; i.1|f) = \max\{p, 0.5\}$, instead of $V(1; i.1|f) = p$. If $p \leq 0.5$, $t = 1$ suffices now as foresight for the agent to choose the middle branch at $i.1$ and expect a strictly better payoff than before.

It is equally straightforward to construct a strictly dominated strategy whose introduction makes the decision-maker better off. Following Nature moving Right at $n.1$ in Figure 1, suppose that the agent can now choose between two moves, say Left and Right, for a payoff of -1 and 0 respectively. The new game is equivalent to the old one for a rational decision-maker. Yet, the continuation payoff of moving Up at $i.1$ is now $-p$, for $t = 1$. The horizon $(1; i.1)$ becomes sufficient for the agent to choose the middle branch at $i.1$. This is strictly beneficial if $p < 0.5$.

This model of decision-making under bounded rationality is clearly affected by strictly dominated strategies, if the foresight is not long enough to reveal the dominance. However, this is more a feature of limited foresight rather than of the particular way continuation values beyond the horizons are modeled here. Similar examples can be constructed for any valuation criterion on the set $\bar{\Omega}_T[t; h|s, f]$, as long as the criterion is strictly increasing with the worst outcome in this set.

3.2 Robustness

The framework used to establish the behavior under limited foresight given in (3) is robust to many different ways of evaluating a given (s, f) with respect to the horizon $(t; h)$. Specifically, the framework can accommodate any functional $U(t; h|s, f)$. We could assign continuation values for the horizon $(t; h)$ under (s, f) using any criterion on the set $\bigcup_{f' \underset{(t;h)}{\approx} f} \bar{\Omega}_T[t; h|s, f]$ of all possible terminal outcomes that may occur beyond the horizon. A different valuation criterion for the continuation problem beyond a horizon $(t; h)$ would define a new value function $\tilde{V}(t; h|f)$. This would, in turn, require a new transformation $\tilde{g}(t; h|f)$ to represent the discrepancies between $\tilde{V}(t; h|f)$ and the criterion of rational decision-making. Nevertheless, the fundamental feature of the approach this paper takes towards modeling bounded rationality would still obtain. A representation is still feasible based on the trade off between $\tilde{V}(t; h|f)$ and the costs $\tilde{C}(t; h|f) = [1 - \tilde{g}(t; h|f)] \tilde{V}(t; h|f)$ of complexity.

Monotonicity, however, holds as a general property only if the continuation value of the horizon $(t; h)$ under (s, q) is strictly increasing with the worst outcome in the set $\bar{\Omega}_T[t; h|s, q]$. This matters since monotonicity is crucial for the most important features of this model. The property allows for the optimal horizon to be obtained endogenously. It also specifies the precise way in which rational decision-making corresponds to a limiting case of the representation. More importantly, it provides the necessary structure for the complexity costs to be inferred from observed choices.

Even more features must be given up if we insist on more general criteria. For example, suppose that the continuation value of the horizon $(t; h)$ were to be dependent upon the entire set $\bigcup_{f' \underset{(t;h)}{\approx} f} \bar{\Omega}_T[t; h|s, f]$ of terminal outcomes that may occur beyond the horizon if the profile (s, f) is played within it. In this case, the benefits and the costs of further search are no longer necessarily zero along dimensions where further search is futile.⁷

3.3 A Specification for the Complexity Costs

I will now present a particular specification for the transformation function g which provides the search for optimal alternatives against f under limited foresight with an *endogenous* stopping rule. Specifically, the optimal pair $\{t_h(f), BR(t_h(f); h|f)\}$ is obtained by considering the nested sequence $\{(t; h)\}_{t=1}^{T_h}$ until the marginal benefit from enlarging the search horizon falls short of the marginal complexity cost of doing so.

Lemma 3 For $t^* \in \mathbf{T}_h \setminus \{T_h\}$, suppose that

$$(i) \Delta_t g(t; h|f) > 0 \quad \forall t < t^*$$

⁷Consider again the game of Figure 2 with f as in Example 2.5. Clearly, there are infinitely many plays f' by Nature that are $(2; i.1)$ -equivalent to f . If $V(2; i.1)$ depends on outcomes in the set $\bigcup_{f' \underset{(t;h)}{\approx} f} \bar{\Omega}_T[t; h|s, f]$, then $\Delta_t V(2; i.1) \neq 0$.

(ii) $\Delta_t g(t; h|f) < 0$ and $V(t; h|f) > -g(t+1; h|f) \frac{\Delta_t V(t; h|f)}{\Delta_t g(t; h|f)}$, $\forall t \geq t^*$

Then, t^* is uniquely optimal in (7).

Proof. Observe that

$$\begin{aligned} \Delta_t [g(t; h|f) V(t; h|f)] &= \Delta_t [V(t; h|f) - C(t; h|f)] \\ &= g(t+1; h|f) \Delta_t V(t; h|f) + V(t; h|f) \Delta_t g(t; h|f) \end{aligned}$$

and recall that (M) implies $\Delta_t V(t; h|f) \geq 0$ for all $t \in \mathbf{T}_h$ while $V(t; h|f) > 0$ for all $(t, h, f) \in \mathbf{T}_h \times H \times \Delta(Q)$.⁸

Take first $t < t^*$. By condition (i) of the Lemma, we get $\Delta_t V(t; h|f) > \Delta_t C(t; h|f)$. For $t \geq t^*$, condition (ii) gives $\Delta_t V(t; h|f) < \Delta_t C(t; h|f)$. ■

This lemma establishes sufficient conditions for the representation of bounded rationality given in (7) to be single-peaked on \mathbf{T}_h . The conditions allow the complexity cost function to depict the extent of bounded rationality as this varies across agents for a given decision problem. Specifically, if the observed responses against f at h by two agents reveal different levels for their respective optimal foresight, then the behavior of the more myopic decision-maker will correspond to higher marginal complexity costs for the values of foresight that lie between the two optima.

Definition 4 Let $\mathcal{C}(S|h, f) = BR(t; h|f)$ and $\mathcal{C}'(S|h, f) = BR(t'; h|f)$ be two boundedly-rational choice rules against $f \in \Delta(Q)$ at h . $\mathcal{C}(S|h, f)$ is **more boundedly-rational** than $\mathcal{C}'(S|h, f)$ against f at h if

$$t < t' \quad \text{and} \quad BR(t; h|f) \neq BR(t'; h|f) \tag{8}$$

The inequality condition along with monotonicity property imply that $V(t; h|f) < V(t'; h|f)$. That is, the optimal responses against f under foresight t correspond to strictly lower expected payoff than those under foresight t' . Intuitively, this definition is based upon the observation that the agent with the more myopic foresight will do worse against f . By the argument made in the proof of Lemma 3, this implies that, if $C(\cdot; h|f)$ and $C'(\cdot; h|f)$ are complexity costs functions for the choice rules $\mathcal{C}(S|h, f)$ and $\mathcal{C}'(S|h, f)$ respectively, then

$$\Delta C'_\tau(\tau; h|f) < \Delta V_\tau(\tau; h|f) < \Delta C_\tau(\tau; h|f) \quad \forall \tau : t \leq \tau < t'$$

In other words, we have the following result:

Claim 1 Suppose that $\mathcal{C}(S|h, f) = BR(t; h|f)$ is a more boundedly-rational choice rule than $\mathcal{C}'(S|h, f) = BR(t'; h|f)$ against f at h ($t < t'$). Suppose also that $C(\cdot; h|f)$ and $C'(\cdot; h|f)$ are the corresponding complexity cost functions and let the conditions of Lemma 3 apply. Then

$$\Delta_\tau C(\tau; h|f) > \Delta_\tau C'(\tau; h|f) \quad \forall \tau : t \leq \tau < t' \tag{9}$$

⁸Recall that all terminal payoffs are strictly positive, $r : \Omega_T \rightarrow \mathbb{R}_{++}$. Clearly, so are the values of $u(h|s, f)$ and $U(t; h|f)$, for any $(s, f) \in S \times \Delta(Q)$, $t \in \mathbf{T}_h$, and any $h \in H$.

As a particular example of a setting where Lemma 3 applies, let $g : \mathbf{T}_h \times \Delta(Q) \rightarrow \mathbb{R}_{++}$ be defined by

$$g(t; h|f) = e^{-k_h(f)[t+1-t_h(f)]^2} \quad k_h(f) \in \mathbb{R}_{++} \quad (10)$$

Then

$$\Delta_t g(t; h|f) = e^{-k_h(f)[t+1-t_h(f)]^2} \left(e^{-k_h(f)[1+2(t+1-t_h(f))]} - 1 \right) = g(t; h|f) \left(e^{-k_h(f)[1+2(t+1-t_h(f))]} - 1 \right)$$

and the two conditions of the lemma, with respect to the transformation function, are satisfied for $t^* = t_h(f) - 1$. Moreover, when $t \geq t_h(f) - 1$, we have

$$\begin{aligned} & \Delta_t [g(t; h|f) V(t; h|f)] \\ &= g(t+1; h|f) V(t+1; h|f) - g(t; h|f) V(t; h|f) \\ &= g(t; h|f) \left[e^{-k_h(f)[1+2(t+1-t_h(f))]} V(t+1; h|f) - V(t; h|f) \right] \\ &< g(t; h|f) \left[e^{-k_h(f)} V(t+1; h|f) - V(t; h|f) \right] \end{aligned}$$

Choosing $k_h(f) > \max_{t \in \{t_h(f)-1, \dots, T_h-1\}} \ln \left(\frac{V(t+1; h|f)}{V(t; h|f)} \right)$, this specification meets the requirements of Lemma 3. Since all terminal payoffs are finite, there clearly exists $k_h(f)$ sufficiently large so that the lemma applies with $t^* = t_h(f) - 1$.⁹

For the specification in (10), $g(t; h|f) \leq 1$ for all $t \in \mathbf{T}_h$. That is, the function C is now a cost function on the entire set \mathbf{T}_h , not only on $\mathbf{T}_h(f)$ as in the general case. Observe also that the optimal foresight of bounded rationality under the specification in (10) is given by

$$t_h(f) = 1 + \min \{t \in \mathbf{T}_h : \Delta_t V(t; h|f) < \Delta_t C(t; h|f)\} \quad (11)$$

The optimal horizon is found by examining the nested sequence $\{(t; h)\}_{t=1}^{T_h}$ until the marginal complexity cost of enlarging the horizon first outweighs the marginal benefit of doing so.¹⁰ This stopping-rule for search can be interpreted as a *satisficing* condition, to use the terminology of Simon [44]. In contrast to popular interpretations, satisficing here is not the phenomenon of giving up when the decision-maker has achieved a predetermined level of utility. As in MacLeod [24], satisficing instead refers to stopping search when it is believed that further search is unlikely to yield a higher net payoff.

⁹I allow for the parameter k depend upon the current information set $h \in H$ and Nature's play $f \in \Delta(Q)$ so that comparisons across members of $H \times \Delta(Q)$ can be made (see the analysis of Section 4.3). When such variation is not important, the specification in (10) can be given for a unique parameter k , for all $(h, f) \in H \times \Delta(Q)$, under the stronger condition: $k > \max_{(h, f) \in H \times \Delta(Q)} \max_{t \in \{t_h(f)-1, \dots, T_h-1\}} \ln \left(\frac{V(t+1; h|f)}{V(t; h|f)} \right)$.

¹⁰This agrees closely with the intuition of Stigler [46] who described the notion of an optimal stopping rule for search of satisfactory alternatives via the example of a person who wants to buy a used car and stops searching when the costs of further search would exceed the benefit from it.

3.4 Empirical Testing

Under the specification in (10), $C(t; h|f) = \alpha V(t; h|f)$ with $\alpha = 1 - g(t; h|f) \in [0, 1)$. Observe that α is *independent* of the terminal payoffs. This independence implies that the benefits and costs of search are related in a way that is particularly useful for the empirical inference of the complexity costs function. Specifically, if the terminal payoffs are increased, the costs of complexity should also increase but not as quickly. In other words, higher benefits to search offer an incentive for the decision-maker to extend the search horizon.

In a given decision problem, suppose that the optimal foresight under bounded rationality is t^* . By (11), it must be $\Delta_t V(t^*; h|f) < \Delta_t C(t^*; h|f)$. Now let the terminal payoffs increase so that $V(t^* + 1; h|f)$ rises while $V(t^*; h|f)$ stays constant. $C(t^*; h|f)$ remains unchanged, and $C(t^* + 1; h|f)$ increases but not as quickly as $V(t^* + 1; h|f)$. Clearly, we will eventually get to the point where $\Delta_t V(t^*; h|f) = \Delta_t C(t^*; h|f)$. By the algorithm in (11), this suffices for the agent to extend her horizon beyond $(t^*; h)$. The analysis of the specification in (10) implies that the critical increase v in $V(t^* + 1; h|f)$ is given by $v = e^{k_h(f)} V(t^*; h|f) - V(t^* + 1; h|f)$.

This suggests a procedure for empirical inference in decision problems where observed choices reveal the limited foresight of the subject. To illustrate, consider the decision-making situation described in Example 2.4 with $p < 0.5$. For an agent who chooses the upper branch at $i.1$, it must be $\Delta_t C(1; i.1|f) > 0.5$. For another who chooses the middle branch, $\Delta_t C(1; i.1|f) \leq 0.5$. If, moreover, the second agent announces also an intention to play Down (Up) at $i.4$, we can further deduce that $\Delta_t C(3; i.1|f) > (\leq) 0.25$. We can proceed now in one of two ways. Starting with the individual who chose the Upper branch at $i.1$, we present her with a series of these games with increasing payoffs at $n.4$, and $i.4$ until the first instant where she chooses the middle option. This would determine $\Delta_t C(1; i.1|f)$ for this individual. Continuing now increasing the payoffs at $i.4$, we will arrive at the first instant where she chooses to move Up at $i.4$. This would determine $\Delta_t C(3; i.1|f)$. Alternatively, we could present the game with a randomly-drawn payoff structure to a population of agents and estimate the lowest $\Delta_t V(1; i.1|f)$, for which the middle option is chosen at $i.1$ and $\Delta_t V(3; i.1|f)$, for which the intention to move Up at $i.4$ is announced at $i.1$ given that the middle branch is chosen at $i.1$. Of course, the model would be falsified if an agent chooses the lower branch at $i.1$ or switches from the middle to the upper branch at $i.1$ while presented with increasing $\Delta_t V(1; i.1|f)$.

Notice that the above are estimates for the costs of bounded rationality implied by the complexity costs model. They would determine the cost-benefit trade-offs in the suggested behavior under limited foresight for it to replicate the observed choices of a subject. In this way, my analysis abstracts from some well-known issues associated with modeling limited search as optimization with decision costs. First, reliable estimates of the true underlying benefits and costs (such as opportunity or computational costs) can demand large degrees of knowledge. Second, the knowledge and the computations involved can be so massive that one is forced to assume that ordinary people

have the computational abilities and statistical software of econometricians (Conlisk [6]; Sargent [39]). My *as if* approach to modeling departures from rational decision-making is based on a simple fact. Whatever goes on inside people’s minds in decision-making situations (what many authors have referred to as the “black box” of bounded rationality), it does result in their choices. These choices reveal a particular cost-benefit relation when they are compared to the behavior under limited foresight presented here. And this is sufficient for a representation.

3.5 Related Literature

An interesting class of limited foresight models (Jehiel [18], [19]) differ from mine in two respects. First, the foresight is fixed at some $t \in \mathbf{T}_h$. Second, the continuation payoff is given by some subjective distribution the agent places on the set of terminal outcomes that may occur beyond $(t; h)$ given that the profile (s, f) is played within the horizon. This approach allows for uncertainty about Nature’s plans of action beyond the horizon. The agent is not expected to know what Nature will do in contingencies where she currently does not know what she will play.

As already argued above, the valuation criterion that these models use for the continuation problem can be accommodated in my framework. This criterion appears to demand less in terms of knowing the possible outcomes beyond the horizon. Yet, this is not necessarily true once we are concerned with the weakest form of consistency in forecasts. If we require the subjective assessment to be consistent with possible plays, it must be equivalent to some (σ', f') such that σ' and f' are $(t; h)$ -equivalent to s and f respectively. This means that we ought to know the entire set $\Omega_T[h|s', q']$ of terminal contingencies, for any (s', q') played with positive probability under (σ', f') .¹¹ In contrast, I require that one knows only the worst outcome in the set $\bar{\Omega}_T[t; h|s, q]$, for every q in the support of f . Each of these outcomes necessarily corresponds to some (pure) play.

A sizeable literature on information theory and computational complexity has avoided the assumption of fixed foresight. Instead, boundedly-rational decision-making in complex problems is modeled via algorithms for searching through game-trees. The central issue of which direction and how far into the tree one should look ahead is usually addressed by various interpretations of the same principle. A search path is evaluated by assigning (i) the value given by the underlying preferences to the terminal nodes that can be reached within the path, (ii) some value for the continuation game beyond the path, and (iii) some computational costs for considering the contingencies encountered by the path. McLeod [24] combines (ii) and (iii) into one heuristic function. He argues that this is a more general approach since decomposing (ii) and (iii) implies that one can costlessly create a probability distribution over uncertain events. This assumption often leads to underestimating the difficulty of a problem. The complexity of a problem is very sensitive to not only the size of the search graph but also the structure of the heuristic function.

The similarities between McLeod’s approach and mine are obvious. A profile (s, f) determines

¹¹The set $\Omega_T[h|s', q']$ is a singleton only for non-repeated decision problems.

the paths, within a horizon $(t; h)$, that play may follow with positive probability. With respect to (ii) above, the valuation criterion I propose requires an estimate for the worst possible outcome beyond these paths. On the one hand, this is stronger than what is required by MacLeod’s heuristic. It presupposes that knowing the worst outcome is costless. On the other hand, it allows for the discrepancies between the criterion and rational choices to be measured by the complexity costs alone. This, in turn, allows the costs to be inferred from observed choices. The heuristic function, as well as its decomposition in (ii) and (iii), depends on two sources of bounded rationality: limited foresight and estimation of the benefits of further search. As in the models of fixed limited foresight, the interdependency of the two hinders inference from observed behavior.

A considerable body of work on bounded rationality has modeled complexity in repeated games via the use of finite automata. These are machines with a finite number of states. Their operating instructions are given by (i) an output function that assigns an action at each state, and (ii) a transition function that assigns a succeeding state to every pair of a current state and an action by the opponent (Neyman [26], Rubinstein [34], Abreu and Rubinstein [1], Piccione and Rubinstein [28], Kalai and Stanford [21]). An automaton is meant to be an abstraction of the process by which a player implements a rule of behavior in a repeated game. The complexity of a machine M is given by $|M|$, the number of its states. A player’s preferences over pairs of machines is increasing in her payoff in the repeated game, and decreasing with the complexity of her machine.

Diasakos [10] shows that, when facing a machine M , the decision-making rule in (3) is equivalent to the one in the implementation complexity model. Intuitively, M defines a plan of action for a finite number of contingencies. This is a strategy $q(M) \in Q$ which repeats this plan cyclically. Each cycle corresponds to a horizon of foresight $t(M) = 2|M|$. By being a cycle, moreover, at the end of the horizon play returns to the initial state. The worst continuation payoff beyond this horizon is $\min_{s' \in S} u(s, q(M))$, which is independent of any $s \in S$ the agent might be using within the horizon. The rule in (3) chooses, thus, an s that responds optimally against $q(M)$ within $(t(M); h)$. Any such s is $(t(M); h)$ -equivalent to the optimal machine against M .

Notice also that the complexity cost approach agrees with most of the “general principles” of decision-making in complex problems given by Geanakoplos and Gray [14]. Considering the nested sequence of horizons $\{(t; h)\}_{t=1}^{T_h}$ guarantees that the *proximity* (thinking about the immediate consequences of your actions first) principle is met, but leads to at least as good choices as the *relevance* principle (examining the most promising alternatives first).¹² Taking into account the worst outcome beyond the horizon leads to a set of options that are considered good rather than singling out one possibility as excellent (the *family* principle). Moreover, it dictates that one is cautious about uncertain future contingencies. This is consistent with the *clear sight* principle

¹²This principle dictates that the path defined by the most promising alternative is followed deeper into the game tree than the remaining alternatives. In my setting, the horizon of search applies the same depth to all alternatives. Following the most promising alternative down a path of length t leads to choosing a response that cannot do better but can do worse than that chosen in my setting under foresight t .

(if possible, one should base decisions on clear-cut results: observations with the lowest error probability should typically receive the most weight). Its extreme cautiousness means, however, that my model does not abide by the *stability* principle (consistency increases confidence so that, if our guesses are accurate, we should guess values along the intended path that change very little).

4 Applications

This section illustrates several important aspects of complexity costs and demonstrates that the model is consistent with three well-known paradoxes in decision-making. These paradoxes obtain in seemingly unrelated decision-making situations for obviously psychologically different reasons. Accordingly, a variety of models and arguments have been suggested in the literature to explain them. Yet, the complexity costs approach provides a common framework for depicting the underlying limitations that might force an agent’s choices to depart from those prescribed by rational decision-making. In each case, paradoxical behavior results when the complexity costs are large relative to the benefits of search, preventing a boundedly-rational agent from considering the problem in its entirety. For the first application, in particular, the model reconciles the findings of two experimental studies that seem to be at odds under other theoretical explanations.

4.1 Temporal Framing

The two problems of Figure 3 differ only in their temporal framing. In Γ_{np} , a random draw takes place first. With probability $1 - p$, the game ends immediately and the decision-maker gets $\$x_3 \geq 0$. Otherwise, the agent is asked to choose between ending the game and receiving $\$x_2$ ($x_2 > x_3$) or allowing for another random draw. In this case, she can get $\$x_1$ ($x_1 > x_2$) with probability q and $\$x_3$, otherwise. The setup of Γ_p is the same but for pre-commitment. The agent must decide, before any resolution of uncertainty, whether or not she will terminate the game if it does not end at the first draw.

Table 1: Temporal Framing (Normal Form)

	UU	UD	DU	DD
D	x_1	x_3	x_3	x_3
E	x_2	x_2	x_3	x_3

Standard economic theory describes two methods by which such decision problems are solved: the strategy method and backward induction. Under the strategy method, the decision-maker looks at the normal form representation of the game, considers her set of possible strategies, calculates the implications of each for the probability distribution over final outcomes, and chooses the optimal strategy with respect to her underlying preferences over final outcomes. The strategy method

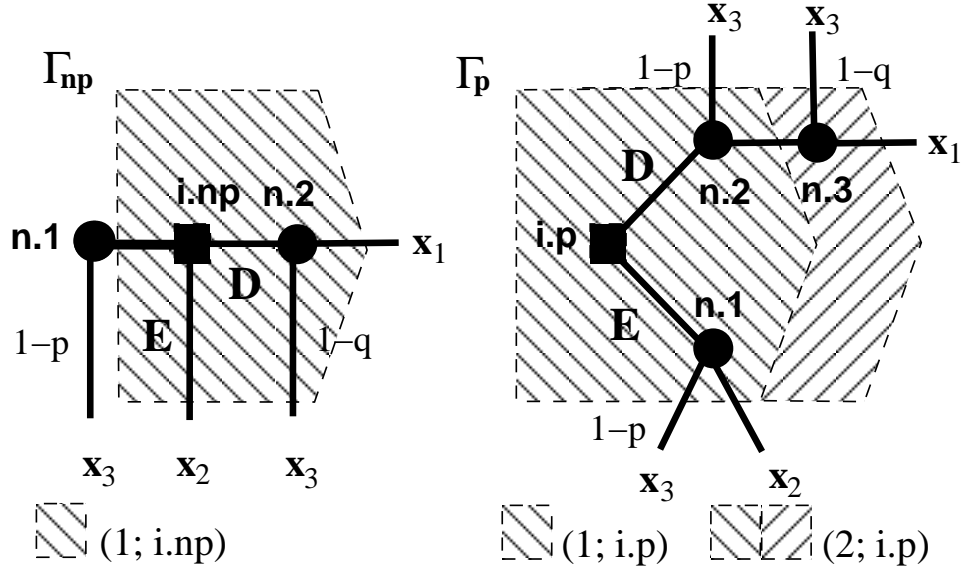


Figure 3: Temporal Framing (Extensive Forms)

views the two problems as identical because they have the same normal form representation. In both problems, the agent chooses only once with two available actions: “Take the Draw” (D) and “End the Game” (E). In both, choosing E versus D corresponds to a difference of $\mathbb{E}[\Delta u(x)] = p[u(x_2) - qu(x_1) - (1 - q)u(x_3)]$ in unconditional ex-ante expected utility. Let $\mathbb{E}[\Delta u(x)] > 0$, so that E is optimal, and denote by f Nature’s play.¹³

In contrast, the complexity cost approach is consistent with different choices between the two problems. In Γ_{np} , the agent does not have to make any decisions prior to the first draw. Starting at $i.np$, the horizon $(1; i.np)$ covers the continuation problem. Complexity costs $C(1; i.np|f)$ suffice, therefore, for her to figure out that choosing between E and D corresponds to an expected utility difference $\frac{1}{p}\mathbb{E}[\Delta u(x)]$. The net expected utility difference from considering this is $\frac{1}{p}\mathbb{E}[\Delta u(x)] - C(1; i.np|f)$.

In problem Γ_p , on the other hand, she has to commit in advance to an action. At $i.p$, it is the horizon $(2; i.p)$ that covers the continuation problem. By Proposition 2, however, enlarging

¹³In Table 1, $Q = \{U, D\} \times \{U, D\}$ where $\{X, Y\} \in Q$ denotes the outcome of X for the first draw and Y for the second. The two independent draws are given by $f = \{pq, p(1 - q), (1 - p)q, (1 - p)(1 - q)\}$.

the horizon is not costly if it introduces information sets where no decisions are made. Thus, $C(2; i.p|f) = C(1; i.p|f)$. The expected utility difference from considering the problem is $\mathbb{E}[\Delta u(x)]$. The net expected utility difference is given by $\mathbb{E}[\Delta u(x)] - C(1; i.p|f)$.

If

$$\mathbb{E}[\Delta u(x)] - C(1; i.p|f) < 0 \leq \frac{1}{p} \mathbb{E}[\Delta u(x)] - C(1; i.np|f) \quad (12)$$

the two problems are not equivalent. In the pre-commitment problem, the complexity costs exceed the expected utility difference between the two available choices. In other words, pre-commitment corresponds to complexity that is more costly than the expected benefit from considering the two choices. Since it is not optimal to incur the costs of complexity, the agent will be unable to figure out the expected utility difference between her two options. She will be indifferent, at $i.p$, between continuing or terminating the game after the first draw. In contrast, incurring the complexity costs is optimal in problem Γ_{np} . She will choose to terminate the game at $i.np$.

Cubitt *et al.* [9] investigated such choices experimentally. They presented their subjects with (non-graphical) descriptions of Γ_{np} , Γ_p , and a third problem Γ_s . In Γ_s , the agent must choose between the simple lottery that follows E and the compound one that follows D, respectively, in the tree depicting Γ_p in Figure 3. Cubitt *et al.* find that play in the first two problems differs in a statistically significant way: up to 70% of their subjects chose E in Γ_{np} and 57% chose D in Γ_p . They did not observe any significant difference in play across the last two problems.¹⁴

Since Nature is a non-strategic player, Γ_s is equivalent in extensive form to Γ_p and, consequently, in normal form to both Γ_{np} and Γ_p . This means that the strategy method cannot account for the findings of the study. Under backward induction, however, the agent looks at the extensive form representation and may not be indifferent across the three problems depending on how the pre-commitment problem Γ_p is evaluated. There are two possibilities. The agent may consider her decision as being made at the beginning of the game, in which case Γ_p is equivalent to Γ_s . Alternatively, she may regard the decision as to be made when she actually gets to play. Now, Γ_p is equivalent to Γ_{np} (regardless of whether backward induction is combined with reduction of compound lotteries or substitution with certainty equivalents).

Nevertheless, whatever her preferences across problems, the standard setting of expected utility optimization dictates that E is always chosen. Cubitt *et al.* turn, therefore, to non-expected utility theories for an explanation of their experimental finding. They consider those of Machina [25], Segal [38], Kahneman and Tversky [20], and Karni and Safra [22]. Except for that of Karni and Safra, all theories are rejected as they require timing independence in decision-making which forces the same predictions for Γ_{np} and Γ_p . In contrast, the behavioral consistency theory of Karni and Safra does not require timing independence. In that model, Γ_p is equivalent to Γ_s , but not to Γ_{np} .

¹⁴Problems 2, 3 and 4 in Cubitt *et al.* [9] correspond to Γ_{np} , Γ_p and Γ_s respectively. Problems 1 and 5 are, respectively, scaled versions of Problems 2 and 4. The authors run their experiments with $x_1 = 16$, $x_2 = 10$, $x_3 = 0$, $p = 0.25$, and $q = 0.8$.

Because it is based on generalized expected utility, the theory permits the common ratio effect to obtain, accommodating different responses in Γ_{np} and Γ_p .¹⁵

An alternative explanation can be supported also by the model of Koszegi and Rabin [23]. An agent can opt for different choice across the two problems if her risk attitudes depend upon her reference point. When facing the pre-commitment problem Γ_p , her reference point can be taken to be her current endowment and the problem involves no potential losses. Choice of E at $i.p$ corresponds to gaining x_2 with probability p and remaining at the reference point with probability $1 - p$. D corresponds to a gain of x_1 with probability pq and remaining at the reference point with probability $1 - pq$. Problem Γ_{np} , on the other hand, does entail the possibility of losses. Choice of E at $i.np$ corresponds to gaining x_2 with certainty and this can be now incorporated into the agent's reference point. The option D corresponds now to a gamble between a gain of $x_1 - x_2$ with probability pq and a loss of $x_2 - x_3$ with probability $1 - pq$.

Explaining the findings of Cubitt *et al.* [9] based on either the common ratio effect or reference-dependence loss aversion seems to be at odds, however, with the findings of another study. Instead of choices between continuing or terminating the game after the first draw, Hey and Paradiso [17] elicit subjects' preferences for the same three problems via their willingness to pay for playing them.¹⁶ Since all valuations are now made ex-ante, the reference point is the same for both problems and they should be viewed as equivalent under reference-dependence loss aversion. Yet, only 56% of the subjects examined by Hay and Paradiso revealed that they were indifferent between the two problems. For an agent who chooses D in Γ_p and E in Γ_{np} , the ex-ante expected value is, respectively, $pqu(x_1) + (1 - pq)u(x_3)$ and $pu(x_2) + (1 - p)u(x_3)$. Under the common ratio effect, Γ_p should be preferred to Γ_{np} but only a third of the subjects were willing to pay a significantly higher amount for playing Γ_p .

The complexity costs approach can accommodate the findings of this study too. For the majority of the subjects of Hey and Paradiso, the complexity costs seem to be trivial and the three problems are equivalent. For any problem, the ex-ante expected value is $pu(x_2) + (1 - p)u(x_3)$. For a third of the subjects, though, the costs are non-trivial but incurring them is optimal. The expected monetary payoff from any problem is $pu(x_2) + (1 - p)u(x_3)$. If $C(1; i.p|f) < C(1; i.np|f)$, however, the net expected value of the pre-commitment problem Γ_p is higher than that of Γ_{np} . This required condition on the complexity costs across the two problems can indeed obtain under the specification in (10). Since $C(t; h|f) \leq V(t; h|f)$, the value of the horizon is an upper bound for the corresponding complexity costs.¹⁷ In problems Γ_{np} and Γ_p , upper bounds for $C(1; i.np|f)$ and $C(1; i.p|f)$ are given by $\frac{1}{p}\mathbb{E}[\Delta u(x)]$ and $\mathbb{E}[\Delta u(x)] < \frac{1}{p}\mathbb{E}[\Delta u(x)]$, respectively.

¹⁵Consider lotteries defined on the set $X = \{x_1, x_2, x_3\}$ of the monetary consequences $x_1 > x_2 > x_3 \geq 0$. A simple prospect can be denoted by the vector $(\pi_1, \pi_2, 1 - \pi_1 - \pi_2)$ of probabilities assigned, respectively, to x_1 , x_2 , and x_3 . The common ratio effect occurs when the option $(0, 1, 0)$ is chosen over $(q, 0, 1 - q)$ while $(pq, 0, 1 - pq)$ is chosen over $(0, p, 1 - p)$, for some $q, p \in (0, 1)$.

¹⁶Hey and Paradiso use $x_1 = 50$, $x_2 = 30$, and $x_3 = 0$. The probabilities p, q are the same as in Cubitt *et al.* [9].

¹⁷Recall that $C(t; h|f) = [1 - g(t; h|f)]V(t; h|f)$, with $g(t; h|f) \in (0, 1]$ for all $t \in \mathbf{T}_h$.

The complexity costs depict the expected utility difference between fully- and boundedly-rational choices. Obviously, the higher the expected benefit from thinking further into a decision problem, the higher ought to be the penalty for not doing so. In this example, the boundedly-rational choice is always to not terminate the game after the first draw. In problem Γ_p , failing at $i.p$ to choose E corresponds to an expected loss of $-p\mathbb{E}[\Delta u(x)]$. In Γ_{np} , on the other hand, failing to choose E at $i.np$ carries an expected loss of $-\mathbb{E}[\Delta u(x)]$. Ex-ante, the expected loss from making the wrong choice is the same in both problems. At the time of making the decision, however, it is higher at $i.np$.

When the entire decision-tree is costlessly considered, the ex-ante value of a problem does not depend upon the timing of decisions. Under limited foresight, though, the expected benefit of further search does. Since the complexity costs measure the discrepancies between the two, the timing of decisions matters for the costs of complexity. Moreover, it matters even when the costs do not prevent the decision-maker from considering the entire problem and choosing the plans of action that are optimal under rational decision-making. The complexity costs approach allows for violations of both timing independence and dynamic consistency in decision-making.¹⁸ The timing of the resolution of uncertainty is important, therefore, beyond the way in which it is when dynamic consistency is required. Under dynamic consistency, the timing of the resolution of uncertainty matters for the ex-ante value of a decision problem only if it affects the choice of optimal plans. Under complexity costs, it matters even when the optimal choice remains the same allowing the approach to reconcile the findings of the two experimental studies.

The pre-commitment problem Γ_p requires essentially thinking at $i.p$ about the strategic position, at $i.np$, when some partial resolution of uncertainty will have occurred. Relative to its ex-ante expected utility payoff, this is a more complex undertaking than just waiting to see in which strategic position one ends up and deciding then, as in problem Γ_{np} . For an agent for whom such complexity is too costly relative to the benefit of thinking about pre-commitment, her choices may differ in the two problems. For an agent for whom the complexity costs are not too high yet non-trivial, the choice will be the same but the ex-ante expected net value of Γ_p can exceed that of Γ_{np} . The latter agent can be viewed as sophisticated about her bounded rationality. She realizes that she may err in her choices and this makes the amalgamation of risks offered by the pre-commitment problem Γ_{np} valuable as it mitigates the expected cost of her errors. She prefers to pre-commit as she knows that erring at $i.np$ will be more costly than it appears to be at $i.p$.

4.2 Diversification Bias

Consumers have been repeatedly found to be more variety-seeking when they plan for future consumption than when they make separate sequential choices preceding each consumption period.

¹⁸It is easy to see how dynamic inconsistencies can obtain here. Let Γ_p be a problem about advanced planning instead of pre-commitment. If (12) holds, the agent cannot decide what her best move should be at $i.p$; she is indifferent between her two actions. When she gets to $i.np$, however, she will prefer to not take the second draw.

This effect was first demonstrated by Simonson [45], who gave students the opportunity to select among six snacks in one of two conditions: (i) they picked one snack at each of three weekly class meetings (sequential choice), (ii) on the first meeting, they selected three snacks to be consumed one per week over the meetings (simultaneous choice). The subjects displayed significantly more variety-seeking under simultaneous choice where 64% chose three different snacks. Under sequential choice, only 9% did so. Simonson suggested that this behavior might be explained by variety-seeking serving as a choice heuristic (when asked to make several choices at once, people tend to diversify). Thaler [47], on the other hand, regards this behavior as resulting from a failure of predicted utility to accurately forecast subsequent experienced utility.

Read and Loewenstein [31] replicated Simonson’s diversification effect (they refer to it as the “diversification bias”) for snacks in several experiments. In one study, conducted on Halloween night, their subjects were young trick-or-treaters who approached two adjacent houses. In one setting, the children were offered a choice between two candies at each house. In another, they were told at the first house they reached to “choose whichever two candy bars you like” (with large piles of both candies being displayed so that the children would not think it rude to take two of the same). The results showed again a strong bias for diversification under simultaneous choice: every child selected one of each candy. In contrast, only 48% of those given sequential choice picked different candies. Evidence of diversification bias has been found also in studies of defined contribution savings plans, supermarket purchases, choices of lottery tickets, and audio tracks (see Read *et al.* [32] for a review).

In what follows, I present an example of the complexity cost model which is consistent with the findings of the experiment by Read and Loewenstein. Although the set-up does not allow for an explanation of why people err in the direction of diversification, it accounts for why they do not diversify in the sequential setting.¹⁹ Consider a decision-maker who must choose one of two snacks, a and b , on each of two consecutive periods, 1 and 2. The objects of choice, therefore, are two-period plans in $X = \{a, b\} \times \{a, b\}$. Let $(x, y) \in X$ denote the plan in which x is chosen in the first period and y in the second. Experiments of diversification bias use objects that are familiar to the subjects so that choice or consumption per se provides no new information to the decision-maker about the product itself. Nevertheless, even without any objective uncertainty about static preferences across the different items, the agent has subjective uncertainty about which plan she would prefer over the dynamic setting she must consider. Let this kind of uncertainty be completely described by a finite state space Ω and a probability distribution μ on Ω . For $((x, y), \omega) \in X \times \Omega$, the state-dependent utility is given by some function $u : X \times \Omega \rightarrow \mathbb{R}_{++}$.

¹⁹In the Read and Loewenstein experiment, the decision problem is presented to subjects in a way that allows for erring in only one dimension - choosing a diversified plan. Bias towards diversification has been demonstrated also in other experiments that allow for making errors along more than one dimensions (see Read *et al.* [32]). The one-dimensional character of Read and Loewenstein, however, allows for a description of the problem by a simple model. This comes without any loss of generality in explaining why choices differ across the two settings.

This example aims at illustrating how the temporal nature of a decision problem matters for observed choices. The ensuing analysis is simplified, therefore, by abstracting from the static preferences across snacks. I will assume that the choice of a particular snack matters only with respect to whether or not it adds variety in consumption across periods.

Assumption 4.2.A

$$\begin{aligned} u(a, b, \omega) &= u(b, a, \omega) \\ u(a, a, \omega) &= u(b, b, \omega) \end{aligned} \quad \forall \omega \in \Omega$$

Notice also that, temporal issues aside, diversification should *not* be ex-ante preferable. Otherwise, the bias towards diversification is not really paradoxical for we could have the following scenario. In the simultaneous-choice setting, the diversified plan is chosen since it is optimal according to the ex-ante expected-utility ranking. Under sequential-choice, choosing $x \in \{a, b\}$ in period 1 defines the period-2 choice set to be $\{(x, a), (x, b)\}$. The choice from this set will be according to the state-dependent utility ranking for the state that obtains in the second period. Diversification would be optimal now at any of the states in $\Omega_0 = \{\tilde{\omega} \in \Omega : u(x, y, \tilde{\omega}) > u(x, x, \tilde{\omega})\}$, for $y \neq x$. Since Ω_0 occurs with probability $\mu(\Omega_0)$, an outside observer would record a diversified plan as chosen always in the simultaneous setting but only $100 \times \mu(\Omega_0)\%$ of the time in the sequential one.

Assumption 4.2.B

$$\mathbb{E}_\Omega [u(x, y, \omega)] < \mathbb{E}_\Omega [u(x, x, \omega)] \quad \forall x, y \in \{a, b\} : x \neq y$$

Under these two assumptions, rational decision-making under expected utility is not consistent with the diversification bias. An *undiversified* plan is chosen always in the simultaneous setting and $100 \times (1 - \mu(\Omega_0))\%$ of the time in the sequential one. However, this is not necessarily the case in the complexity costs model. The temporal framing of the decision problem matters for choices, by affecting the choice set of the agent and, consequently, whether or not dynamic inconsistency matters for final outcomes.

Under sequential choice, opting for x in period 1 leaves $\{(x, a), (x, b)\}$ as the feasible set for period 2. This allows for any dynamically inconsistent choice of plan, made in period 1 (due to complexity being too costly), to be reversed in period 2. Under simultaneous choice, opting for (x, y) in period 1 makes it the singleton feasible set for period 2. The diversification bias can obtain because the choice of plan made in period 1 is now irreversible.

Consider first the simultaneous choice problem depicted by Figure 4. The period-2 realizations of the states in Ω are represented by the terminal nodes $\{i.\omega : \omega \in \Omega\}$. In period 1, the agent has to choose and commit to one of the four plans in X . At $i.0$, when thinking about the possible terminal contingencies in Ω , she faces two paths. Each has length $t = 2$ and is associated with the period-2 realization of one of the states in Ω , following a period-1 choice of snack. Accounting for

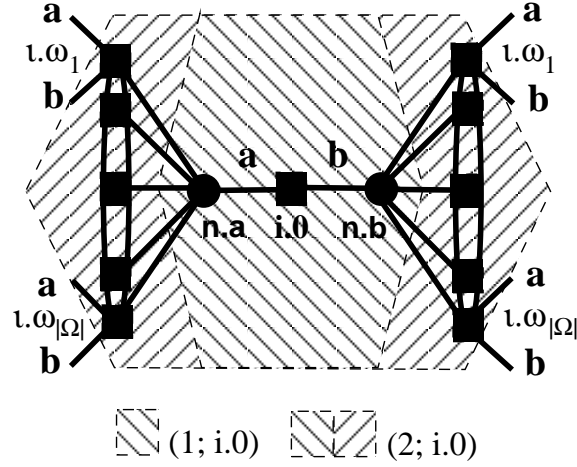


Figure 4: Simultaneous Choice

all terminal contingencies requires, therefore, complexity costs $C(2; i.0|\mu)$. Since there is a unique prior about the terminal nodes, the decision-making rule reduces to that of the standard expected utility ranking between the four plans across the states in Ω . The value of the horizon $(2; i.0)$ is given by

$$V(2; i.0|\mu) = \max_{(x,y) \in X} \sum_{\omega \in \Omega} \mu(\omega) u(x, y, \omega) = \max_{(x,y) \in X} \mathbb{E}_{\Omega} [u(x, y, \omega)]$$

The decision-maker will choose either of the two *undiversified* plans.

Suppose, however, that the agent limits her foresight to $t = 1$. The corresponding complexity cost is $C(1; i.0|\mu)$. Since the horizon does not reach the terminal nodes, the decision problem becomes

$$V(1; i.0|\mu) = \max_{x \in \{a,b\}} \min_{y \in \{a,b\}} \sum_{\omega \in \Omega} \mu(\omega) u(x, y, \omega) = \min_{(x,y) \in X} \mathbb{E}_{\Omega} [u(x, y, \omega)]$$

She will choose either of the two *diversified* plans. In other words, the following condition

$$\Delta_t C(1; i.0|\mu) > \mathbb{E}_{\Omega} [u(x, x, \omega)] - \mathbb{E}_{\Omega} [u(x, y, \omega)] \quad \{x, y\} \in X : x \neq y \quad (13)$$

suffices for the agent to choose a diversified plan when she has to commit to a decision about a two-period plan at the beginning of period 1.

Under sequential choice, if $x \in \{a, b\}$ was picked in period 1, the period-2 choice will be from $\{(x, b), (x, a)\}$. The decision-tree is the same as that of Figure 4, albeit for each state $i.\omega$ following either from $n.a$ or $n.b$ defining a singleton information set. At each $i.\omega$, the decision problem is trivial in terms of complexity costs: $V(i.\omega|\mu) = \max_{y \in \{a, b\}} u(x, y, \omega)$. At $i.0$, though, the problem is similar to that of Figure 4 in terms of complexity. There are two horizons the agent might consider: $(1; i.0)$, and $(2; i.0)$. The corresponding complexity costs $C(1; i.0|\mu)$, $C(2; i.0|\mu)$ and the values $V(1; i.0|\mu)$, $V(2; i.0|\mu)$ are the same as before.

If (13) holds, the agent will again find it not worthwhile, at $i.0$, to think about her period-2 choice once she has chosen something in period 1. She will choose either of the two diversified plans. If (13) does not hold, on the other hand, she will choose either of the two undiversified ones. Nevertheless, a choice of a plan $(x, y) \in X$, at $i.0$, is now merely an *intention* to pick y in period 2, not a commitment. Having chosen x in period 1, when she chooses again at $i.\omega$, it will be out of the set $\{(x, a), (x, b)\}$. Picking $(x, y) : y \neq x$ will then be optimal at any state in Ω_0 . Since this occurs with probability $\mu(\Omega_0)$, an outside observer would record a diversified plan only $100\mu(\Omega_0)\%$ of the time.

When (13) holds, the predictions of the model are in agreement with the observed bias towards diversification. Under simultaneous choice, at the beginning of period 1, the agent must commit to a plan which will irrevocably determine her utility for both subsequent periods. Clearly, there is an advantage in considering the entire time-horizon of the problem. If the corresponding complexity is too costly, however, diversification results.

Under sequential choice, at the time of making the period-2 decision, the uncertainty about preferences will have been resolved. Thinking about this resolution of uncertainty in period 1 is still a complex undertaking but the agent is not penalized for failing to do so. If the complexity costs are too high so that (13) applies, she will have the wrong intentions in period 1 about period-2 play. Yet, this can be corrected by period-2 actual decisions. Diversification as a final outcome obtains only in the states of Ω_0 , when it is actually called for.

4.3 Risk Aversion in the Small and Large

This application presents an intertemporal model of consumption and savings in which larger amounts of wealth to be allocated across time induce longer planning horizons. Such a relation between the wealth and the length of the optimal horizon has significant implications for risk attitudes in the small and large. Here, I show that the model can accommodate plausible but contrasting preferences between small- and large-stakes gambles that cannot be explained under standard settings.

Consider a decision-maker who has an endowment w_0 , and a choice between accepting or not a lottery \tilde{z} which would result in her wealth being $\tilde{w}_0 = w_0 + \tilde{z}$. Rabin [29] and Safra and Segal [37] show that, under expected as well as non-expected utility maximization, rejecting certain small

gambles implies that ridiculously favorable gambles should also be turned down. More precisely, let $V(\tilde{w})$ denote the value functional for wealth prospects \tilde{w} . If $V(\tilde{w}'_0) \leq V(w_0)$ for a small-stakes gamble \tilde{w}'_0 , then there exist very favorable gambles \tilde{w}''_0 for which it is also the case that $V(\tilde{w}''_0) < V(w_0)$. In fact, this is the case for gambles \tilde{w}''_0 involving small losses, very large winnings, and winning probabilities such that very few people, if any, would reject them. Such predictions contradict plausible patterns of behavior since many people would turn down the small gambles.

Suppose, however, that the agent also faces the problem of allocating her wealth for consumption and savings across a lifetime of T periods. Under the complexity costs approach, her optimal horizon, $t(\tilde{w}) \in \{1, \dots, T\}$, for the temporal allocation of any wealth prospect \tilde{w} enters into the value function $V(t(\tilde{w}); \tilde{w})$. The example presented below establishes that an increasing $t(\cdot)$ can be consistent with rejecting small gambles while accepting sufficiently favorable large gambles.

Rabin [29] offers a strong argument for why expected utility theory is unable to provide a plausible account of risk aversion over modest stakes. Turning down a modest-stakes gamble means that the marginal utility of wealth diminishes quickly even for small changes in wealth. His examples indicate that seemingly innocuous risk aversion with respect to small gambles implies an absurd rate of decrease in the marginal utility of wealth. Safra and Segal [37] extend this criticism to several non-expected utility theories.

With complexity costs, two competing effects govern choices: the decreasing marginal utility of wealth, and the fact that a larger wealth level leads to a jump in the value of wealth when it induces a longer optimal horizon. The value of any amount of wealth is higher when this amount gets allocated over a longer horizon, because it is better allocated. The issue that Rabin, Safra and Segal point out applies, other things being equal, within a fixed horizon. The following example suggests that (i) it can be outweighed when the value of money increases sufficiently with the length of the optimal horizon and (ii) the increase in the value of money from extending the horizon increases with the amount of wealth to be allocated.

Apart from being consistent with contrasting risk attitudes in the small and large, an increasing relation between wealth and the length of the optimal horizon also explains why risk aversion would be, in particular, large in the small but small in the large. The possible realizations of a lottery define different contingencies with respect to the temporal wealth-allocation problem. Gambles with small stakes correspond to contingencies in which the resulting wealth levels are similar enough to the endowment to be allocated under the same horizon as the endowment. When the stakes are large, however, the contingency of winning big motivates the agent to think further into the future. A longer planning horizon means that a larger fraction of the wealth endowment is allocated optimally and, thus, used to evaluate the prospect.

With non-increasing relative risk aversion utility for wealth, this results in risk aversion being smaller in the large than in the small. Moreover, it can produce realistically high degrees of risk aversion in the small even when the endowment is too large for this to obtain under standard settings. This is the case when the complexity costs of allocating wealth are high enough for the

agent to be sufficiently myopic in her planning relative to her lifetime. Then, the corresponding part of her lifetime endowment that is used for optimal allocation is small, and her risk aversion large.

In the complexity costs model, differences in the corresponding optimal horizons can produce sharp contrasts in the way agents evaluate risky prospects in the small and large. When comparing very large winnings versus very small losses, this can also account for acceptance of gambles with very large winnings and very small losses, even when the probability of winning is extremely low. In other words, it can explain why people purchase lottery tickets. In the standard settings, accepting bets with extremely low expected gains requires convex utility functions for wealth. In the presence of complexity costs, this can result from players' attitudes towards lotto-type bets being more sensitive to the large jackpot than to the minuscule probability of winning. The case of lotto-type bets is examined in Appendix A.2. In what follows, I present a simplified version that focuses on risk aversion in the small and large.

Formally, consider an agent who will live for 3 periods, denoted by $\tau \in \{1, 2, 3\}$ (the general, T -period version is analyzed in Appendix A.1). She has a wealth endowment w_0 , and must allocate her wealth for consumption across her lifetime. Her preferences are represented by a time-separable function, with utility for consumption in period τ given by $u_\tau(c_\tau) = \log c_\tau$. I will assume also that she has a belief regarding the worst possible level of consumption in period τ , given by $\underline{c}_\tau > 0$.²⁰ To complete the description, any wealth not consumed in period $\tau \in \{1, 2\}$ can be invested in a safe asset that generates a period- τ rate of return r_τ . The discount factor between two consecutive periods is $\delta \in (0, 1)$.

Suppose the agent is considering a horizon of length $t \in \{1, 2, 3\}$. Her decision problem for allocating any amount of wealth w is given by

$$V(t; w) = \max_{\mathbf{c} \in \mathbf{C}} \left\{ \sum_{\tau=1}^t \delta^{\tau-1} \log c_\tau + \min_{\mathbf{c}' \in \mathbf{C}: \mathbf{c}' \stackrel{\tau}{\simeq} \mathbf{c}} \sum_{\tau=t+1}^3 \log c'_\tau \right\}$$

where \mathbf{C} denotes the set of feasible lifetime consumption plans. These are plans $\mathbf{c} = \{c_\tau\}_{\tau=1}^3$ satisfying the following system of constraints:

$$s_1 = w - c_1, \quad s_{\tau'} = s_{\tau'-1}(1 + r_{\tau'-1}) - c_{\tau'} \quad \tau' \geq 2, \quad s_\tau \geq 0, \quad c_\tau \geq \underline{c}_\tau \quad \tau \in \{1, 2, 3\} \quad (14)$$

where s_τ denotes the wealth available at the beginning of period $\tau + 1$. Clearly, this requires that the initial wealth suffices for the plan $\{\underline{c}_\tau\}_{\tau=1}^3$ of least consumption:²¹

$$w = \sum_{\tau=1}^3 \underline{c}_\tau \prod_{i=1}^{\tau-1} (1 + r_i)^{-1} \quad (15)$$

²⁰An obvious interpretation of such a belief is the requirement for some subsistence level of consumption.

²¹Abusing notation slightly, in what follows, the term $\prod_{i=1}^0 x_i$ will denote the number 1, for any $x_i > 0$.

With respect to a horizon of length t , two lifetime consumption plans \mathbf{c}, \mathbf{c}' are equivalent ($\mathbf{c} \simeq_t \mathbf{c}'$) iff $c_\tau = c'_\tau$ for all $\tau \leq t$. The worst outcome for the remainder of her life beyond the horizon would obtain if she were to consume at $\{\underline{c}_\tau\}_{\tau=t+1}^3$. Therefore, for a horizon of length t , the decision problem can be re-written as follows

$$V(t; w) = \max_{\mathbf{c} \in \mathbf{C}} \left\{ \sum_{\tau=1}^t \delta^{\tau-1} \log c_\tau + \sum_{\tau=t+1}^3 \delta^\tau \log \underline{c}_\tau \right\}$$

For $\mathbf{c} \in \mathbf{C}$, let $\mathbf{c}^t = \{c_\tau\}_{\tau=1}^t$ denote the restriction of \mathbf{c} to the horizon of length t . If the agent chooses the consumption plan \mathbf{c}^t within the horizon, her available wealth at the beginning of period $t+1$ will be

$$s_{t+1}(\mathbf{c}^t) = w \prod_{i=1}^t (1+r_i) - \sum_{\tau=1}^t c_\tau \prod_{i=\tau}^t (1+r_i) \quad t \in \{1, 2\}$$

Along with the system of constraints in (14), this requires the following constraint to be met by any solution $\mathbf{c} \in \mathbf{C}$

$$w = \sum_{\tau=1}^3 c_\tau \prod_{i=1}^{\tau-1} (1+r_i)^{-1} \quad (16)$$

Notice also that, since $V(2; w) = V(3; w)$, rational decision-making requires $t \geq 2$.²²

Let the agent consider a horizon of length $t = 1$. Given that she anticipates her consumption beyond the horizon to be at the worst level, the choice of period-1 consumption is given trivially by the budget constraint (16):

$$c_1 = w - \left(\frac{1}{1+r_1} \right) \left(\underline{c}_2 + \frac{\underline{c}_3}{1+r_2} \right)$$

The value of the horizon is

$$V(1; w) = \log \left[w - \left(\frac{1}{1+r_1} \right) \left(\underline{c}_2 + \frac{\underline{c}_3}{1+r_2} \right) \right] + \delta \log \underline{c}_2 + \delta^2 \log \underline{c}_3$$

For $t \geq 2$, the problem becomes a standard one. The optimal consumption vector is $c_1^* = \frac{w}{1+\delta+\delta^2}$, $c_2^* = \delta c_1^* (1+r_1)$, and $c_3^* = \delta^2 c_1^* (1+r_1)(1+r_2)$. The value of the problem is

$$\begin{aligned} V(2; w) &= (1+\delta+\delta^2) \log w + \delta \log [\delta(1+r_1)] \\ &\quad - (1+\delta+\delta^2) \log [1+\delta+\delta^2] + \delta^2 \log [\delta^2(1+r_1)(1+r_2)] \end{aligned}$$

²²For $t = 2$, a given plan $\{c_1, c_2\}$ chosen for consumption within the horizon corresponds to a unique level of consumption in the final period as determined by (16). Planning for the first two periods implicitly accounts also for consumption in the last period.

Consider now the agent facing a lottery \tilde{z} , which would result in her wealth being $\tilde{w}_0 = w_0 + \tilde{z}$. For simplicity, I will restrict attention to 50-50 lose $\$l$ /gain $\$g$ lotteries ($g > l$), denoted by $(g, \frac{1}{2}; -l, \frac{1}{2})$. Under such gambles, the expected values of the horizons are given by

$$\begin{aligned} & \mathbb{E}_{\tilde{w}_0} [V(1; \tilde{w}_0)] \\ &= \frac{1}{2} \left(\log \left[w_0 + g - \left(\frac{1}{1+r_1} \right) \left(c_2 + \frac{c_3}{1+r_2} \right) \right] + \log \left[w_0 - l - \left(\frac{1}{1+r_1} \right) \left(c_2 + \frac{c_3}{1+r_2} \right) \right] \right) \\ & \quad + \delta (\log c_2 + \delta \log c_3) \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}_{\tilde{w}_0} [V(2; \tilde{w}_0)] &= \frac{(1 + \delta + \delta^2)}{2} \log [(w_0 + g)(w_0 - l)] + \delta \log [\delta(1 + r_1)] \\ & \quad - (1 + \delta + \delta^2) \log [1 + \delta + \delta^2] + \delta^2 \log [\delta^2(1 + r_1)(1 + r_2)] \end{aligned}$$

Let $t(\tilde{w}_0)$ and $t(w_0)$ denote the corresponding optimal horizons for the wealth allocation problem under the lottery \tilde{w}_0 and the endowment, respectively. The agent will accept the lottery if and only if $\mathbb{E}_{\tilde{w}_0} [V(t(\tilde{w}_0); \tilde{w}_0)] > V(t(w_0); w_0)$. That is, the relative lengths $t(\tilde{w}_0)$ and $t(w_0)$ of the optimal horizons matter for the decision of whether or not to accept the lottery. The following two claims establish precisely how they matter.

Claim 2 *For a gamble $(g, \frac{1}{2}; -l, \frac{1}{2})$, suppose that the induced optimal horizon is not longer than that induced under the certain prospect, $t(\tilde{w}_0) \leq t(w_0)$. If*

$$w_0 - \mathbf{1}_{t(\tilde{w}_0)=1} \left(\frac{1}{1+r_1} \right) \left(c_2 + \frac{c_3}{1+r_2} \right) \leq \frac{gl}{g-l} \quad (17)$$

then the gamble will be rejected. For $t(\tilde{w}_0) = t(w_0)$, $(g, \frac{1}{2}; -l, \frac{1}{2})$ is rejected only if (17) holds.

Proof. Observe first that $\mathbb{E}_{\tilde{w}_0} [V(t; \tilde{w}_0)]$ and $V(t; w_0)$ have all, but their respective first, terms equal. If both problems induce fully-rational planning, $t(\tilde{w}_0) > 1$, the claim is immediate by comparing $\mathbb{E}_{\tilde{w}_0} [V(2; \tilde{w}_0)]$ and $V(2; w_0)$. Otherwise, we have

$$\begin{aligned} & \left(w_0 + g - \left(\frac{1}{1+r_1} \right) \left(c_2 + \frac{c_3}{1+r_2} \right) \right) \left(w_0 - l - \left(\frac{1}{1+r_1} \right) \left(c_2 + \frac{c_3}{1+r_2} \right) \right) \\ &= \left(w_0 - \left(\frac{1}{1+r_1} \right) \left(c_2 + \frac{c_3}{1+r_2} \right) \right)^2 + (g-l) \left(w_0 - \left(\frac{1}{1+r_1} \right) \left(c_2 + \frac{c_3}{1+r_2} \right) \right) - gl \end{aligned}$$

By (17), this implies $\mathbb{E}_{\tilde{w}_0} [V(t(\tilde{w}_0); \tilde{w}_0)] \leq V(t(\tilde{w}_0); w_0)$. Since $t(\tilde{w}_0) \leq t(w_0)$, the monotonicity property (M) gives $V(t(\tilde{w}_0); w_0) \leq V(t(w_0); w_0)$. ■

Rational planning requires a horizon of length $t \geq 2$. Clearly, a gamble being rejected under the standard setting is a stronger condition than (17).

Corollary 1 Suppose that a gamble $(g, \frac{1}{2}; -l, \frac{1}{2})$ is rejected under the standard setting:

$$w_0 \leq \frac{gl}{g-l} \quad (18)$$

Then, it will be turned down in the complexity costs model, whatever the length of the horizon.

Claim 3 For a gamble $(g, \frac{1}{2}; -l, \frac{1}{2})$, suppose that it induces a longer optimal horizon than the certain prospect, $t(\tilde{w}) > t(w_0)$. If

$$(w_0 + g)(w_0 - l) > Z(\delta)^2 \left(w_0 - \left(\frac{1}{1+r_1} \right) \left(\underline{c}_2 + \frac{\underline{c}_3}{1+r_2} \right) \right)^2 \quad (19)$$

where

$$Z(\delta) = \left(1 + \delta + \frac{1}{\delta} \right)^{\delta+\delta^2} \left(\frac{1 + \delta + \delta^2}{\delta^{\delta^2}} \right)$$

then $(g, \frac{1}{2}; l, \frac{1}{2})$ will be accepted.

Proof. For the gamble to be accepted, it must be $\mathbb{E}_{\tilde{w}_0} [V(2; \tilde{w}_0)] > V(1; w_0)$. By (16), for either realization of the gamble, we have $\tilde{w}_0 > \frac{\underline{c}_2}{1+r_1}, \frac{\underline{c}_3}{(1+r_1)(1+r_2)}$. In other words,

$$\begin{aligned} \left(\frac{\delta + \delta^2}{2} \right) \log [(\tilde{w}_0 + g)(\tilde{w}_0 - l)] &> \delta \log \underline{c}_2 + \delta^2 \log \underline{c}_3 \\ &\quad - \delta \log [1 + r_1] - \delta^2 \log [(1 + r_1)(1 + r_2)] \end{aligned}$$

It suffices, therefore, to have

$$\begin{aligned} &\log [(w_0 + g)(w_0 - l)] + 2 [\delta \log \delta - (1 + \delta + \delta^2) \log [1 + \delta + \delta^2] + \delta^2 \log \delta^2] \\ &> \log \left[w_0 - \left(\frac{1}{1+r_1} \right) \left(\underline{c}_2 + \frac{\underline{c}_3}{1+r_2} \right) \right]^2 \end{aligned}$$

which is equivalent to (19). ■

For complexity costs to account for different risk attitudes in the small and large, it must be that small gambles correspond to shorter optimal horizons than large ones. In what follows, I present sufficient conditions for this to be the case. Specifically, I establish conditions that suffice for the optimal horizon induced by the certain prospect w_0 to be at least as long as the one for a small gamble but shorter than that for a large one. To facilitate comparisons of optimal foresight, observe that

$$\begin{aligned} &2 (\mathbb{E}_{\tilde{w}_0} [\Delta_t V(1; \tilde{w}_0)] - \Delta_t V(1; w_0)) \\ &= 2 \mathbb{E}_{\tilde{w}_0} [V(2; \tilde{w}_0) - V(1; \tilde{w}_0)] - 2 [V(2; w_0) - V(1; w_0)] \\ &= (1 + \delta + \delta^2) \log \left[\frac{(w_0 + g)(w_0 - l)}{w_0^2} \right] \\ &\quad - \log \left[\frac{\left(w_0 + g - \left(\frac{1}{1+r_1} \right) \left(\underline{c}_2 + \frac{\underline{c}_3}{1+r_2} \right) \right) \left(w_0 - l - \left(\frac{1}{1+r_1} \right) \left(\underline{c}_2 + \frac{\underline{c}_3}{1+r_2} \right) \right)}{\left(w_0 - \left(\frac{1}{1+r_1} \right) \left(\underline{c}_2 + \frac{\underline{c}_3}{1+r_2} \right) \right)^2} \right] \quad (20) \end{aligned}$$

Before proceeding, I must specify what are the costs of complexity in the context of this application as well as what gambles should be regarded as small in the presence of complexity costs. The complexity is that of the decision problem of allocating any given amount of wealth w for consumption and savings across the agent's lifetime. For a horizon $t \in \{1, 2, 3\}$ in the allocation problem, the complexity costs will be given by some function $C(t; w)$.

When considering a gamble, the agent must address the allocation problem for each amount of wealth resulting from the possible realizations of the gamble. In this context, it is natural to regard as small any gamble such that all of its possible realizations are small enough for the problem of allocating the corresponding wealth to not differ, in terms of complexity, from that of allocating the endowment.²³

Definition 5 *A possible realization z of a lottery \tilde{z} is a **small stake** if*

$$C(t; w_0 + z) = C(t; w_0) \quad t \in \{1, 2\}$$

*A lottery \tilde{z} is **small** if every z in the support of \tilde{z} is a small stake:*

$$C(t; w_0 + \tilde{z}) = C(t; w_0) \quad t \in \{1, 2\}$$

This definition requires that the complexity cost function is constant, with respect to wealth and given the length t of the horizon, over some range $(w_0 - M, w_0 + M) : M > 0$, around the endowment level: $|z| \leq M$ for a small stake. Equivalently,

$$g(t; w_0 + \tilde{z}) = 1 - \frac{[1 - g(t; w_0)] V(t; w_0)}{\mathbb{E}_{w_0 + \tilde{z}}[V(t; w_0 + \tilde{z})]} \quad (21)$$

for $t \in \{1, 2\}$ and for a small gamble \tilde{z} .

By (20), the following condition

$$\begin{aligned} \left(\frac{w_0^2}{(w_0 + g')(w_0 - l')} \right)^{1+\delta+\delta^2} &\geq \\ &\frac{\left(w_0 - \left(\frac{1}{1+r_1} \right) \left(c_2 + \frac{c_3}{1+r_2} \right) \right)^2}{\left(w_0 + g' - \left(\frac{1}{1+r_1} \right) \left(c_2 + \frac{c_3}{1+r_2} \right) \right) \left(w_0 - l' - \left(\frac{1}{1+r_1} \right) \left(c_2 + \frac{c_3}{1+r_2} \right) \right)} \end{aligned} \quad (22)$$

is equivalent to $\mathbb{E}_{\tilde{w}'_0} [\Delta_t V(1; \tilde{w}'_0)] \leq \Delta_t V(1; w_0)$.²⁴ Under the preceding definition, (22) suffices for

$$\mathbb{E}_{\tilde{w}'_0} [\Delta_t V(1; \tilde{w}'_0)] - \Delta_t C(1; \tilde{w}'_0) \leq \Delta_t V(1; w_0) - \Delta_t C(1; w_0) \quad (23)$$

²³The definition does not specify a relation for $t = 3$ as this would be redundant. Recall that $t = 2$ is sufficient for rational planning: $V(2; w) = V(3; w)$, for any wealth realization w . By Proposition 2, $C(2; w) = C(3; w)$.

²⁴Let us denote by $x = \left(\frac{1}{1+r_1} \right) \left(c_2 + \frac{c_3}{1+r_2} \right)$ the present value of the worst possible future consumption stream. For any g', l' , the right-hand side of (22) tends to 0^- as $x \rightarrow w_0$. The left-hand side does not depend on x and remains positive and bounded away from zero for small gambles ($g', l' \leq M$). That is, for any small gamble $(g', \frac{1}{2}; -l', \frac{1}{2})$, (22) will be valid as long as x is sufficiently close to the initial endowment w_0 .

to hold and, therefore, for the algorithm in (11) to give $t(\tilde{w}'_0) \leq t(w_0)$, for a small gamble $(g', \frac{1}{2}; -l', \frac{1}{2})$.²⁵ In this case, Claim 1 establishes a sufficient condition for the small gamble to be turned down.

Let us now turn to large gambles. Since the focus of this application is on explaining why very favorable gambles could be accepted even though small gambles are rejected, I restrict attention to gambles offering large gains but small losses; that is, to gambles $(g'', \frac{1}{2}; -l'', \frac{1}{2}) : g'' > l''$, where l'' is a small stake but g'' is not. The complexity costs argument requires that large gambles induce longer optimal horizons than small ones. If (22) holds for a small gamble, it suffices that the optimal horizon under the large gamble corresponds to rational planning when the one induced under the endowment does not, $t(w_0) = 1$. For this, it is required that the expected marginal benefit of extending the horizon from $t = 1$ to $t = 2$ under the large gamble exceeds the corresponding marginal benefit under the certain prospect.

For $\mathbb{E}_{\tilde{w}''_0} [\Delta_t V(1; \tilde{w}''_0)] > \Delta_t V(1; w_0)$, it suffices that the following inequality holds

$$\left(\frac{w_0^{1+\delta+\delta^2}}{w_0 - \left(\frac{1}{1+r_1}\right) \left(c_2 + \frac{c_3}{1+r_2}\right)} \right)^2 < \frac{((w_0 + g'')(w_0 - l''))^{1+\delta+\delta^2}}{\left(w_0 + g'' - \left(\frac{1}{1+r_1}\right) \left(c_2 + \frac{c_3}{1+r_2}\right)\right) \left(w_0 - l'' - \left(\frac{1}{1+r_1}\right) \left(c_2 + \frac{c_3}{1+r_2}\right)\right)} \quad (24)$$

In this inequality, only the right-hand side depends upon the realizations of the gamble. Since $\frac{(w_0 - l'')^{1+\delta+\delta^2}}{w_0 - l'' - \left(\frac{1}{1+r_1}\right) \left(c_2 + \frac{c_3}{1+r_2}\right)} > (w_0 - l'')^{\delta+\delta^2}$ and the gamble involves small losses, $l'' \leq M$, the right-hand side of (24) is bounded below by the quantity $\underline{G} = \frac{(w_0 + g'')(w_0 - l'')(w_0 - M)^{\delta+\delta^2}}{w_0 + g'' - \left(\frac{1}{1+r_1}\right) \left(c_2 + \frac{c_3}{1+r_2}\right)}$. Yet the gains are not small and $\lim_{g'' \rightarrow +\infty} \underline{G} = +\infty$. Hence, (24) will be satisfied for gambles with small losses but sufficiently large gains.

The preceding discussion established that, for sufficiently large winnings, the marginal benefit of enlarging the horizon under the gamble exceeds the corresponding marginal benefit under the certain prospect. Therefore, if large enough winning stakes also give²⁶

$$\mathbb{E}_{\tilde{w}''_0} [\Delta_t V(1; \tilde{w}''_0)] > \Delta_t C(1; \tilde{w}''_0) \quad (25)$$

they will induce rational planning, $t(\tilde{w}''_0) = 2$.²⁷ In this case, Claim 3 gives a sufficient condition for the large gamble to be accepted.

²⁵For a small gamble $(g', \frac{1}{2}; -l', \frac{1}{2})$, we have $\Delta_t C(t; \tilde{w}'_0) = \Delta_t C(t; w_0)$, $t \in \{1, 2\}$. Hence, (23) reduces to $\mathbb{E}_{\tilde{w}'_0} [\Delta_t V(1; \tilde{w}'_0)] \leq \Delta_t V(1; w_0)$.

²⁶For example, consider the following conditions: $\Delta_t g(1; \tilde{w}''_0) = \frac{V(1; w_0) \Delta_t g(1; w_0)}{\mathbb{E}_{\tilde{w}''_0} [V(1; \tilde{w}''_0)]}$ and $g(2; \tilde{w}''_0) \geq g(2; w_0)$. Clearly, $\mathbb{E}_{\tilde{w}''_0} [\Delta_t V(1; \tilde{w}''_0)] > \Delta_t V(1; w_0)$ suffices for $\Delta_t [g(1; \tilde{w}''_0) \mathbb{E}_{\tilde{w}''_0} [V(1; \tilde{w}''_0)]] > \Delta_t [g(1; w_0) V(1; w_0)]$ (see the proof of Lemma 3).

²⁷It should be pointed out that (25) is concerned with the *levels* of utility in absolute terms, not relative to wealth. This matters since it could be argued that the smaller the wealth of the agent, the more important it is that she

Claims 2 and 3 allow for a large favorable gamble to be accepted when a small one is turned down. To illustrate, consider an initial endowment w_0 and a complexity cost function $C(t; \cdot) : t \in \{1, 2, 3\}$ such that the problem of allocating w_0 for consumption and savings across one's lifetime is sufficiently complex to induce myopic planning: $\Delta_t V(1; w_0) < \Delta_t C(1; w_0)$. Let $(g', \frac{1}{2}; -l', \frac{1}{2})$ be a small gamble that satisfies (22). It induces, therefore, the same myopic horizon as the certain prospect, $t(\tilde{w}'_0) = t(w_0) = 1$. If (17) holds, Claim 2 dictates that this small gamble ought to be rejected.

Let also $(g'', \frac{1}{2}; -l'', \frac{1}{2})$ be a gamble with small losses but sufficiently large gains for (24) to hold. If, moreover, (25) is satisfied, the large gamble induces rational planning, $t(\tilde{w}''_0) = 2$. Claim 3 provides a sufficient condition, (19), for the large gamble to be accepted. Since (17) holds, an even stronger acceptance condition is given by

$$(w_0 + g'')(w_0 - l'') > Z(\delta)^2 \left(\frac{g'l''}{g' - l'} \right)^2 \quad (26)$$

Observe that $G = 3^6 \left(\frac{g'l''}{g' - l'} \right)^2$ is an upper bound for the quantity on the right-hand side of (26).²⁸

Rabin [29] presents values for g'' , l'' , g' , l' , and w_0 such that many people would turn down the small bet but few - if any - would reject the large one (see Tables I and II, and the discussion pp. 1282-1285). For example, let $g'' = 635,670$, $l'' = 4,000$, $g' = 105$, $l' = 100$, and $w_0 = 340,000$ or $g'' = 36 \times 10^9$, $l'' = 600$, $g' = 125$, $l' = 100$, and $w_0 = 290,000$. Observe that (27) holds in both cases. If (17) obtains for the small gamble, it will be rejected under the horizon of length $t = 1$ and so will be the large gamble. However, (26) is satisfied in both cases. The corresponding upper bounds for its right-hand side are $G = 32.15 \times 10^8$ and $G = 1.8 \times 10^8$ whereas the values for its left-hand side are given by 32.78×10^{10} and 1.04×10^{16} respectively. Hence, the large gambles will be accepted under the horizon of length $t = 2$. Similar results can be obtained for all large bets in Rabin's Table II.²⁹

For a small gamble to be turned down, condition (17) requires that the part of the endowment w_0 which is allocated optimally under the myopic horizon, $w_0 - \left(\frac{1}{1+r_1} \right) \left(c_2 + \frac{c_3}{1+r_2} \right)$, is small. This is in contrast to the standard expected log-utility setting, which requires the entire endowment to

allocates it optimally, and hence, that her horizon is long. This is an argument, however, as to why the expected net (of complexity costs) marginal benefit of further planning *relative* to wealth should be falling with wealth.

²⁸The quantity $Z(\delta)$ was defined in the proof of Claim 2. We have

$$\begin{aligned} \log Z(\delta) &= (1 + \delta + \delta^2) \log [1 + \delta + \delta^2] - (\delta + 2\delta^2) \log \delta \\ &< (1 + \delta + \delta^2) (\log [1 + \delta + \delta^2] - \log \delta) = (1 + \delta + \delta^2) \log [1 + \delta + \delta^{-1}] \end{aligned}$$

where the inequality follows from $\delta \in (0, 1)$. It is trivial to check that the last quantity above is increasing in δ . Clearly, $3 \log 3$ is an upper bound for $\log Z(\delta)$.

²⁹All bets in that Table qualify for the sufficient condition. Yet, only the ones with gains large enough to induce long optimal horizons should be taken in consideration. Notice also that, with respect to Table I, my predictions hold for large gambles and for sufficiently large levels of initial wealth. For example, with $w_0 < 20,000$, large bets will be turned down even though they may induce longer horizons than small bets.

be small. For the two small gambles above to be rejected, it must be that not more than \$2100 and \$500, respectively, are allocated optimally. Of course, these amounts may be unrealistically small. Nevertheless, the two sides of the acceptance condition (26) differ by several orders of magnitude. Thus, the examples do make the point that the model would allow, in general, considerable room for maneuver in the selection of gambles that produce the desired preferences.

Another possible response to Rabin's argument would be that decision-makers do not reject small lotteries even when their expected payoff is almost zero. My analysis abstracts from this issue by assuming sufficient conditions so that small gambles are rejected. This allows the exposition to focus on the essence of Rabin's criticism of expected utility: the rapid deterioration in the value of wealth revealed by rejection of small gambles. For the log-utility specification, this is highlighted by the following condition

$$\frac{g''l''}{g'' - l''} > \frac{g'l'}{g' - l'} \quad (27)$$

For any w_0 , Claim 2 ensures that, once $(g', \frac{1}{2}; -l', \frac{1}{2})$ is rejected over a horizon of length $t \in \{1, 2\}$, any gamble $(g'', \frac{1}{2}; -l'', \frac{1}{2})$ satisfying (27) will also be turned down over the same horizon, no matter how favorable. The above numerical example described a case where (27) holds and both gambles are rejected under myopic planning ($t = 1$) but the large gamble is ultimately accepted if it induces a longer optimal horizon. The example used the acceptance condition (26) which is not compatible with the small gamble being rejected under the standard setting when (27) holds. The acceptance condition (19) of Claim 3, however, is weaker than (26) allowing for $w_0^2 > (w_0 + g'')(w_0 - l'') > Z(\delta)^2 \left(w_0 - \left(\frac{1}{1+r_1} \right) \left(c_2 + \frac{c_3}{1+r_2} \right) \right)^2$. The model can also accommodate pairs of gambles such that both are turned down under the standard setting. By Corollary 1, both gambles will also be rejected under myopic planning. Yet, the latter gamble will be accepted if it induces an optimal horizon of length $t = 2$.

In the complexity costs model, under the myopic horizon $t = 1$, the amount of wealth the agent is using to evaluate risky prospects consists of the part, $w_0 - \left(\frac{1}{1+r_1} \right) \left(c_2 + \frac{c_3}{1+r_2} \right)$, of her endowment that she intends to spend within her horizon. The analysis indicates that the smaller this part, other things remaining unchanged, the more likely that small gambles will be turned down and large gambles will be accepted - see conditions (17) and (19). That is, the model offers a qualitative prediction regarding the heterogeneity of risk attitudes in the small and large across a population of agents: controlling for the agents' initial endowments, *the more optimistic is one regarding her worst possible future consumption stream, the more likely that her risk attitudes will differ at the small and large*. This generalizes to the T -period version of the model where, for given initial endowment w_0 , the more myopic the agent's planning horizon relative to her lifetime, the more likely that conditions (32), (33) and (37) of Appendix A.1 will hold.

Appendix A.2 extends the model to lotto-type large gambles with very large winnings, very small losses and minuscule winning probabilities. I show that, for sufficiently large amounts of wealth, the resulting jump in welfare from extending the allocation horizon outweighs the deterioration in

the marginal value of wealth even in the face of extremely small probabilities of winning. As in the preceding example, my argument depends on the assumption that considering how to allocate one's wealth for consumption in the winning contingency corresponds to a marginal benefit from extending the allocation horizon that outweighs the marginal complexity costs, inducing a longer optimal horizon than that under the endowment. The intuition that accepting lotto-type gambles is fundamentally based upon inducing one to think how to allocate the winnings seems to be in agreement with common marketing practises for such products. Large-prize lotteries are almost always marketed by pointing out various ways the prize could be spent. Advertisements usually describe luxury homes, boats, vacations and retirement locations that could be afforded with the winnings before inviting one to play. Actually, in exactly this spirit of getting one thinking about the contingency of winning, the National Lottery Organization of Greece ran a very successful advertising campaign in 2004-05 with the slogan "But if you win?".

The prediction given above remains valid for the analysis of Appendix A.2: the complexity costs model asserts that, other things being equal, more myopic agents are more likely to accept lotto-type bets. Moreover, in the case of lotto-type bets, the model also predicts that, *poorer agents (with smaller initial endowments) are more likely to accept lotto-type bets*, other things remaining unchanged (see conditions (40), (41) and the subsequent discussion). Observe also that (19) becomes easier to satisfy as the gains g'' increase while losses remain small. This result, that higher potential winnings make it more likely that the bet will be taken other things remaining unchanged, holds also for the lotto-type gambles. Cook and Clotfelter [7] show that across states in the US, lottery-ticket sales are strongly correlated with the size of the state's population, which is in turn positively correlated with the size of the jackpot. Within a state, moreover, ticket sales each week are strongly positively correlated with the size of the rollover.

A natural conclusion from Rabin's criticism is that expected utility should be replaced with more general theories, especially those that exhibit first order risk aversion, those based on uncertainty aversion or those derived from preferences over gains and losses rather than final wealth levels. As observed first by Samuelson [40], expected-utility theory makes also the powerful prediction that agents don't see an amalgamation of independent gambles as significant insurance against the risk of those gambles. He showed that, under expected utility, if (for some sufficiently wide range of initial wealth levels) someone turns down a particular gamble, she should also turn down an offer to play $n > 1$ independent plays of this gamble. In his example, if Samuelson's colleague is unwilling to accept a $(100, \frac{1}{2}; -200, \frac{1}{2})$ bet, he should be unwilling to accept 100 of those taken together. Samuelson's paradox is weaker than Rabin's but it makes the point that adding together a lot of independent risks should not alter attitudes towards those risks in any meaningful way for an expected-utility maximizer. Although rather cumbersome mathematically, it is easy to replace the large 50-50 gamble here by n small ones taken together. For large enough n , this is again a large stakes bet and it will not be turned down if it induces a sufficiently long planning horizon.

Rabin [29] and Rabin and Thaler [30] argue that the way to resolve these paradoxes is loss

aversion. The calibration results of Safra and Segal [37], however, apply to a large collection of non-expected utility theories while both the Rabin and the Samuelson paradoxes can be reframed to apply to pure gains.³⁰ In other words, there seems to be a fundamental discrepancy in the way agents evaluate risky prospects in the small and large, irrespective of the underlying preferences over final outcomes. Indeed, the problems with assuming that risk attitudes over modest and large stakes derive from the same utility function relate to a long-standing debate in economics. Here, the discrepancy arises using the same, simplistic underlying preferences in the small and large. Instead, my argument for why an agent considers a different objective function to evaluate small and large risky prospects derives from considering different horizons for the allocation problem. This results purely from the interaction between the costs and benefits of further search. As long as complexity costs are present, it would apply whatever the valuation functional $V(t; w)$ for the underlying preferences over final wealth outcomes.

Another example of treating differently small and large bets in an intertemporal consumption-savings problem is suggested by Fudenberg and Levine [12]. They model a “dual-self” agent, consisting of a patient long-run self with infinitely-long time-horizon, and a sequence of myopic short-run selves who live for only one period. The long-run self wants to restrain the myopic self from overspending and a commitment device is available in the form of a cash-in-advance constraint. That is, the long-run self decides on how much “pocket cash” to allow the short-run self to have for allocation between consumption and savings during the period. Differences in risk attitudes for small and large bets arise due to a wedge between the propensity to consume out of pocket cash and out of wealth. Winnings from sufficiently small gambles are spent in their entirety, and so are evaluated according to the short-run self’s preferences. But when the stakes are large, self-restraint kicks in because part of the winnings will be saved and spread over the lifetime. Since small and large gambles are evaluated relative to pocket money and wealth, respectively, under constant relative risk aversion, the “dual-self” decision-maker appears to be less risk averse when facing large gambles than when facing small ones.

This intuition is close to the one arising from the complexity costs. Both approaches attribute differences in risk attitudes to differences in the amount of wealth used as reference to evaluate a given gamble. For small gambles, this is restricted to be pocket-cash and $w_0 - \left(\frac{1}{1+r_1}\right) \left(c_2 + \frac{c_3}{1+r_2}\right)$ respectively. As with the pocket-cash interpretation, the smaller is the fraction of the endowment allocated optimally under the certain prospect, the more likely that large bets are accepted even though small ones are not. Equivalently, the more myopic the agent’s planning in allocating her endowment, the more pronounced the divergence in her risk attitudes in the small and large.

³⁰For the Samuelson paradox, consider an agent who (over a sufficiently wide range of initial wealth levels) turns down a 50-50 win \$300 or nothing bet for the certain prospect of \$100. Then she should also prefer \$100n to n plays of the gamble. But as n increases, the expected payoff from the risky prospect becomes infinitely larger than that of the certain one while the probability of winning more by choosing it approaches 1. For the Rabin paradox, if an agent prefers the certain prospect of \$s to a 50-50 bet of winning \$g or nothing, she should also choose $\$ \frac{s^2}{g-2s}$ over a 50-50 bet of winning infinity or nothing.

Chetty and Szeidl [5] offer another interesting analysis of risk preferences. Their model has two consumption goods, one of which involves a commitment in the sense that an adjustment cost must be paid whenever the per-period quantity consumed is changed. Consumption commitments affect risk preferences by amplifying risk aversion against moderate-scale shocks and by creating a certain gambling motive. When the agent cannot adjust commitment consumption easily, small shocks to wealth must be handled by changing adjustable consumption. In the face of a small loss, the marginal utility of wealth increases more quickly when the agent has commitments. The gambling motive obtains when the expected decrease in utility from the adjustable good in the case of a loss is offset by the large increase in utility from the change in commitment consumption that would be afforded under the gain, even though this may occur with small probability.

Their analysis offers a convincing rationalization of why people buy insurance against moderate shocks, change their consumption patterns significantly in the face of unemployment shocks when these lead to large income losses but not when they lead to small ones, or why wealthier individuals are less likely to play the lottery. An important limitation, however, is approximate risk neutrality for small stakes (see their Table II). High risk aversion over moderate stakes offers an explanation for why moderate-stakes gambles would be rejected. It does not explain why risk attitudes should differ in the small and large for a given wealth endowment. Moreover, the gambling motive obtains only for small gambles and for initial wealth levels around the points where the agent is indifferent between adjusting or not commitment consumption.³¹ Rather than explaining why large-stakes gambles would be accepted, this suggests that small gambles shouldn't be rejected.

5 Concluding Remarks

This paper developed a model of endogenous bounded rationality due to search costs arising from the complexity of the decision problem. By inferring the costs of search from revealed preferences through agents' choices, bounded rationality and its extent emerge endogenously. The model thus encompasses endogenous variation across decision-makers and across decision problems in revealed bounded rationality. Under additional assumptions, calibration of search costs suggests predictions and testable implications of the model. Applications to seemingly disparate problems illustrate the flexibility of this endogenous approach. Endogenous complexity costs can be consistent with violations of timing independence in temporal framing problems, dynamic inconsistency and diversification bias in sequential versus simultaneous choice problems, and with plausible but contrasting risk attitudes across small- and large-stakes gambles.

The obvious next step would be to extend this approach to strategic games. Current work in progress establishes existence of equilibrium (in the standard sense of no player having an incentive to deviate when her conjectures happen to be correct) for extensive-form games when players are

³¹See their Proposition 1(iii). Since the value function is locally convex only at these points, the wealth outcomes need to remain within the corresponding neighborhoods.

boundedly-rational. Equilibrium is problematic with currently unforeseen contingencies, however. Instead, I focus on determining which conjectures are justifiable given limited foresight. The fact that the degree of limited foresight obtains endogenously implies a relation between choice of horizon and justifiable conjectures. This relation can be viewed as the bounded rationality analogue to rationalizability. The model can explain some game-theoretical paradoxes in which observed or even intuitively correct responses are at odds with the predictions of backward induction.

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A Appendix

A.1 Risk Aversion in the Small and Large

Here, the analysis of section 3.3 is extended to the general T -period case. For a horizon of length $t \in \{1, \dots, T\}$, the problem of allocating an amount of wealth w becomes

$$V(t; w) = \max_{\mathbf{c} \in \mathbf{C}} \left\{ \sum_{\tau=1}^t \delta^{\tau-1} \log c_{\tau} + \min_{\mathbf{c}' \in \mathbf{C}: \mathbf{c}' \underset{t}{\sim} \mathbf{c}} \sum_{\tau=t+1}^T \delta^{\tau-1} \log c'_{\tau} \right\}$$

where \mathbf{C} denotes the set of lifetime consumption plans $\mathbf{c} = \{c_{\tau}\}_{\tau=1}^T$ satisfying

$$s_1 = w - c_1, \quad s_{\tau'} = s_{\tau'-1}(1 + r_{\tau'-1}) - c_{\tau'} \quad \tau' \geq 2, \quad s_{\tau} \geq 0, \quad c_{\tau} \geq \underline{c}_{\tau} \quad \tau \in \{1, \dots, T\} \quad (28)$$

where s_{τ} denotes the wealth available at the beginning of period $\tau + 1$. Clearly, this requires that the initial wealth suffices for the plan $\{\underline{c}_{\tau}\}_{\tau=1}^T$ of least consumption:

$$w = \sum_{\tau=1}^T \underline{c}_{\tau} \prod_{i=1}^{\tau-1} (1 + r_i)^{-1} \quad (29)$$

With respect to a horizon of length t , the worst case scenario beyond the horizon would obtain if the agent were to consume $\{\underline{c}_{\tau}\}_{\tau=t+1}^T$. Therefore, for a horizon of length t , the decision problem can be re-written as follows

$$V(t; w) = \max_{\mathbf{c} \in \mathbf{C}} \left\{ \sum_{\tau=1}^t \delta^{\tau-1} \log c_{\tau} + \sum_{\tau=t+1}^T \delta^{\tau} \log \underline{c}_{\tau} \right\}$$

For $\mathbf{c} \in \mathbf{C}$, recall that $\mathbf{c}^t = \{c_{\tau}\}_{\tau=1}^t$ denotes the restriction of \mathbf{c} to the horizon of length t . If the agent chooses the consumption plan \mathbf{c}^t within the horizon, her available wealth at the beginning of period $t + 1$ will be

$$s_{t+1}(\mathbf{c}^t) = w \prod_{i=1}^t (1 + r_i) - \sum_{\tau=1}^t c_{\tau} \prod_{i=\tau}^t (1 + r_i) \quad t \in \{1, \dots, T - 1\}$$

Along with the system of constraints in (28), this requires the following constraint to be met by any solution $\mathbf{c} \in \mathbf{C}$

$$w = \sum_{\tau=1}^T c_{\tau} \prod_{i=1}^{\tau-1} (1 + r_i)^{-1} \quad (30)$$

Clearly, rational planning requires a horizon of length $t \geq T - 1$.

Let the agent consider a horizon of length t . Given that she anticipates her consumption beyond the horizon to be at the worst level, $\{\underline{c}_{\tau}\}_{\tau=t+1}^T$, the planning problem can be re-written as follows:

$$V(t; w) = \max_{\mathbf{c}^t \in \mathbf{C}} \left\{ \sum_{\tau=1}^t \delta^{\tau-1} \log c_{\tau} + \sum_{\tau=t+1}^T \delta^{\tau-1} \log \underline{c}_{\tau} \right\}$$

The first-order conditions are given by

$$\frac{c_\tau}{c_1} = \delta^{\tau-1} \prod_{i=1}^{\tau-1} (1 + r_i) \quad 1 < \tau \leq t \quad (31)$$

The optimal vector $\mathbf{c}^*(w) = \left\{ \{c_\tau^*(w)\}_{\tau=1}^t, \{\underline{c}_\tau\}_{\tau=t+1}^T \right\}$ solves the system of t equations defined by (28) and (31) giving

$$c_1^*(w) = \left(\sum_{\tau=1}^t \delta^{\tau-1} \right)^{-1} \left(w - \sum_{\tau=t+1}^T \frac{\underline{c}_\tau}{\prod_{i=1}^{\tau-1} (1 + r_i)} \right)$$

The value of the horizon of length t is given by³²

$$\begin{aligned} V(t; w) &= \sum_{\tau=1}^t \delta^{\tau-1} \log \left(w - \sum_{\tau=t+1}^T \frac{\underline{c}_\tau}{\prod_{i=1}^{\tau-1} (1 + r_i)} \right) + \sum_{\tau=t+1}^T \delta^{\tau-1} \log \underline{c}_\tau \\ &\quad - \sum_{\tau=1}^t \delta^{\tau-1} \log \left(\sum_{\tau=1}^t \delta^{\tau-1} \right) + \sum_{\tau=2}^t \delta^{\tau-1} \log \left(\delta^{\tau-1} \prod_{i=1}^{\tau-1} (1 + r_i) \right) \end{aligned}$$

For a 50-50 lose $\$l$ /gain $\$g$ gamble ($g > l$), we get

$$\begin{aligned} &\mathbb{E}_{\tilde{w}_0} [V(t; \tilde{w}_0)] \\ &= \sum_{\tau=1}^t \frac{\delta^{\tau-1}}{2} \log \left[\left(w + g - \sum_{\tau=t+1}^T \frac{\underline{c}_\tau}{\prod_{i=1}^{\tau-1} (1 + r_i)} \right) \left(w - l - \sum_{\tau=t+1}^T \frac{\underline{c}_\tau}{\prod_{i=1}^{\tau-1} (1 + r_i)} \right) \right] \\ &\quad + \zeta \left(\delta, \{r_i\}_{i=1}^{t-1}, \{\underline{c}_i\}_{i=t+1}^T \right) \end{aligned}$$

where

$$\begin{aligned} &\zeta \left(\delta, \{r_i\}_{i=1}^{t-1}, \{\underline{c}_i\}_{i=t+1}^T \right) \\ &= \sum_{\tau=t+1}^T \delta^{\tau-1} \log \underline{c}_\tau - \sum_{\tau=1}^t \delta^{\tau-1} \log \left(\sum_{\tau=1}^t \delta^{\tau-1} \right) + \sum_{\tau=2}^t \delta^{\tau-1} \log \delta^{\tau-1} + \sum_{\tau=2}^t \delta^{\tau-1} \log \left(\prod_{i=1}^{\tau-1} (1 + r_i) \right) \end{aligned}$$

Recall that $t(\tilde{w}_0)$ and $t(w_0)$ denote the corresponding optimal horizons for the wealth allocation problem under the lottery \tilde{w}_0 and the endowment respectively. The agent will accept the lottery if and only if $\mathbb{E}_{\tilde{w}_0} [V(t(\tilde{w}_0); \tilde{w}_0)] > V(t(w_0); w_0)$. That is, the relative lengths $t(\tilde{w}_0)$ and $t(w_0)$ of the optimal horizons matter for the decision of whether or not to accept the lottery. The following two claims establish precisely how they matter.

³²Abusing notation slightly, the term $\sum_{\tau=2}^1 x_i$ is to be taken as zero, for any $x_i > 0$.

Claim 4 For a gamble $(g, \frac{1}{2}; -l, \frac{1}{2})$, suppose that the induced optimal horizon does not exceed that induced under the certain prospect, $t(\tilde{w}_0) \leq t(w_0)$. If

$$w_0 - \mathbf{1}_{t(\tilde{w}_0) < T-1} \sum_{\tau=t(w_0)+1}^T \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)} \leq \frac{gl}{g-l} \quad (32)$$

then the gamble is rejected. For $t(\tilde{w}_0) = t(w_0)$, the gamble is rejected only if (32) holds.

Proof. $\mathbb{E}_{\tilde{w}_0} [V(t; \tilde{w}_0)]$ and $V(t; w_0)$ have all but the respective first summations equal. If both problems induce rational planning, $t(\tilde{w}_0) \geq T-1$, the Claim is immediate by comparing $\mathbb{E}_{\tilde{w}_0} [V(T-1; \tilde{w}_0)]$ and $V(T-1; \tilde{w}_0)$. Otherwise, notice that

$$\begin{aligned} & \left(w_0 + g - \sum_{\tau=t+1}^T \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)^{-1}} \right) \left(w_0 - l - \sum_{\tau=t+1}^T \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)^{-1}} \right) \\ &= \left(w_0 - \sum_{\tau=t+1}^T \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)^{-1}} \right)^2 + (g-l) \left(w_0 - \sum_{\tau=t+1}^T \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)^{-1}} \right) - gl \end{aligned}$$

By (32), this implies $V(t(\tilde{w}_0); w_0) \geq \mathbb{E}_{\tilde{w}_0} [V(t(\tilde{w}_0); \tilde{w}_0)]$. Since $t(\tilde{w}_0) \leq t(w_0)$, the monotonicity property gives $V(t(w_0); w_0) \geq V(t(\tilde{w}_0); w_0)$. ■

Rational planning requires a horizon of length $t \geq T-1$. Clearly, a gamble being rejected under the standard setting is a stronger condition than (32). In other words, Corollary 1 of Section 3.3 applies also for the T -period version.

Claim 5 For a gamble $(g, \frac{1}{2}; -l, \frac{1}{2})$, suppose that it induces a longer optimal horizon than the certain prospect, $t(\tilde{w}_0) = t(w_0) + k : 1 \leq k \leq T - t(w_0)$. If

$$\begin{aligned} & \exp\left(-\frac{2\zeta_k(\delta, t(w_0))}{\sum_{\tau=1}^{t(w_0)} \delta^{\tau-1}}\right) \left(w_0 - \sum_{\tau=t(w_0)+1}^T \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)} \right)^2 \\ & < \left(w_0 + g - \sum_{\tau=t(w_0)+k+1}^T \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)} \right) \left(w_0 - l - \sum_{\tau=t(w_0)+k+1}^T \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)} \right) \end{aligned} \quad (33)$$

where

$$\zeta_k(\delta, t) = \sum_{\tau=1}^t \delta^{\tau-1} \log\left(\frac{\sum_{\tau=1}^t \delta^{\tau-1}}{\sum_{\tau=1}^{t+k} \delta^{\tau-1}}\right) + \sum_{\tau=t+1}^{t+k} \delta^{\tau-1} \log\left(\frac{\delta^{\tau-1}}{\sum_{\tau=1}^{t+k} \delta^{\tau-1}}\right)$$

then the gamble will be accepted.

Proof. We want $\mathbb{E}_{\tilde{w}_0} [V(t(w_0) + k; \tilde{w}_0)] > V(t(w_0); w_0)$. By (29), we have

$$\begin{aligned}
& \left(w_0 + g - \sum_{\tau=t(w_0)+k+1}^T \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)} \right) \left(w_0 - l - \sum_{\tau=t(w_0)+k+1}^T \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)} \right) \\
& \geq \left(\sum_{\tau=t+1}^{t(w_0)+k} \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)} \right)^2 \\
& = \zeta \left(\delta, \{r_i\}_{i=1}^{t(w_0)+k-1}, \{c_i\}_{i=t(w_0)+k+1}^T \right) - \zeta \left(\delta, \{r_i\}_{i=1}^{t(w_0)-1}, \{c_i\}_{i=t(w_0)+1}^T \right) + \zeta_k(\delta, t(w_0))
\end{aligned}$$

The result follows immediately. ■

For complexity costs to account for different risk attitudes in the small and large, it must be that small gambles correspond to shorter optimal horizons than large ones. In what follows, I present sufficient conditions for this to be the case. Specifically, I establish conditions that suffice for the optimal horizon induced by the certain prospect w_0 to be at least as long as the one for a small gamble but shorter than that for a large one. To facilitate comparisons of optimal foresight, consider

$$\begin{aligned}
& \mathbb{E}_{\tilde{w}_0} \Delta_t V(t; \tilde{w}_0) - \Delta_t V(t; w_0) \\
& = \sum_{\tau=1}^t \frac{\delta^{\tau-1}}{2} \Delta_t \left(\begin{array}{l} \log \left(w_0 + g - \sum_{\tau=t+1}^T c_\tau \prod_{i=1}^{\tau-1} (1+r_i)^{-1} \right) \\ + \log \left(w_0 - l - \sum_{\tau=t+1}^T c_\tau \prod_{i=1}^{\tau-1} (1+r_i)^{-1} \right) \\ - \log \left(w_0 - \sum_{\tau=t+1}^T c_\tau \prod_{i=1}^{\tau-1} (1+r_i)^{-1} \right)^2 \end{array} \right) \\
& + \frac{\delta^t}{2} \left(\begin{array}{l} \log \left(w_0 + g - \sum_{\tau=t+2}^T c_\tau \prod_{i=1}^{\tau-1} (1+r_i)^{-1} \right) \\ + \log \left(w_0 - l - \sum_{\tau=t+2}^T c_\tau \prod_{i=1}^{\tau-1} (1+r_i)^{-1} \right) \\ - \log \left(w_0 - \sum_{\tau=t+2}^T c_\tau \prod_{i=1}^{\tau-1} (1+r_i)^{-1} \right)^2 \end{array} \right)
\end{aligned} \tag{34}$$

As in Section 3.3, small gambles have gains and losses small enough so that the problem of allocating the corresponding wealth to not differ, in terms of complexity, from that of allocating the endowment.

Definition 6 A possible realization z of a lottery \tilde{z} is a **small stake** if

$$C(t; w_0 + z) = C(t; w_0) \quad t \in \{1, \dots, T-1\}$$

A lottery \tilde{z} is **small** if every z in the support of \tilde{z} is a small stake.

For small stakes z , $C(t; \cdot)$ is taken to be constant for some range, $(w_0 - M, w_0 + M) : M > 0$, around the endowment level: $|z| \leq M$.³³ Equivalently, (21) is valid for $t \in \{1, \dots, T-1\}$ and for a small gamble \tilde{z} .

To conserve space, let the present value of the least-consumption stream $\{c_\tau\}_{\tau=t+k}^T$ be denoted by $x_{t+k} = \sum_{\tau=t+k}^T c_\tau \prod_{i=1}^{\tau-1} (1+r_i)^{-1}$. Define also the function $F : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ by $F(g, l, x) = \frac{(w_0-x)^2}{(w_0+g-x)(w_0-l-x)}$. From (34), the following condition

$$\delta^t \log F(g, l, x_{t+2}) \geq \sum_{\tau=1}^t \delta^{\tau-1} \log \left(\frac{F(g, l, x_{t+1})}{F(g, l, x_{t+2})} \right) \quad (35)$$

suffices for $\mathbb{E}_{\tilde{w}_0} \Delta_t V(t; \tilde{w}_0) \leq \Delta_t V(t; w_0)$. Under the preceding definition, (35) is sufficient for the algorithm in (11) to give $t(\tilde{w}'_0) \leq t(w_0)$, for a small gamble $(g', \frac{1}{2}; -l', \frac{1}{2})$.³⁴ In this case, Claim 3 establishes a sufficient condition for the small gamble to be turned down.

Let us now turn to large gambles. Since the focus of this application is on explaining why very favorable gambles could be accepted even though small gambles are rejected, I restrict attention to gambles offering large gains but small losses. In other words, to gambles $(g'', \frac{1}{2}; -l'', \frac{1}{2}) : g'' > l''$, where l'' is a small stake but g'' is not.

The complexity costs argument requires that large gambles induce longer optimal horizons than small ones. If (35) holds for a small gamble, it suffices that the optimal horizon under the large gamble is longer than the one induced under the endowment. For this, it is required that the expected marginal benefit of extending the horizon, under the large gamble, exceeds the corresponding marginal benefit under the certain prospect.

Of course, the analysis should also accommodate large gambles that are rejected under the standard setting. That is,

$$\frac{g'' l''}{g'' - l''} > w_0 \quad (36)$$

For $\mathbb{E}_{\tilde{w}''_0} [\Delta_t V(t; \tilde{w}''_0)] > \Delta_t V(t; w_0)$, it suffices that the following inequality holds

$$\delta^t \log F(g'', l'', x_{t+2}) < \sum_{\tau=1}^t \delta^{\tau-1} \log \left(\frac{F(g'', l'', x_{t+1})}{F(g'', l'', x_{t+2})} \right)$$

Since the losses l'' are bounded above by M but the gains are not, the left-hand side of this inequality tends to $-\infty$ as $g'' \rightarrow +\infty$. In contrast, the right-hand side is bounded below by zero

³³The definition does not specify a relation for $t = T$ as this would be redundant. Recall that $t = T-1$ is sufficient for rational planning: $V(T-1; w) = V(T; w)$, for any wealth realization w . By Proposition 2, $C(T-1; w) = C(T; w)$.

³⁴For any given small stakes g, l , the right-hand side of (35) tends to 0 as $x_{t+1} \rightarrow 0$ (recall that $x_{t+1} > x_{t+2}$). However, if the small gamble is rejected under the standard setting, $F(g, l, \cdot)$ is strictly increasing (see the following Footnote) and, thus, the left-hand side of (35) remains bounded below away from zero. As another argument for why (35) will not be vacuous, notice that it is satisfied as an equality for $g, l \rightarrow 0$. That is, it will hold for sufficiently small gambles.

since $F(g'', l'', \cdot)$ is strictly increasing under (36).³⁵ Thus, the condition will be satisfied for gambles with sufficiently large gains.

Therefore, if large enough gambles give

$$\mathbb{E}_{\tilde{w}_0''} [\Delta_t V(t(w_0); \tilde{w}_0'')] > \Delta_t C(t(w_0); \tilde{w}_0'')$$

they will induce optimal horizons that are longer than those for the certain prospect.³⁶ For simplicity of exposition, suppose that they induce rational planning, $\Delta_t C(T-2; \tilde{w}_0'') < \mathbb{E}_{\tilde{w}_0''} [\Delta_t V(T-2; \tilde{w}_0'')]$. For $k = T - t(w_0) - 1$, (33) reads

$$g'' - l'' \geq \frac{g'' l'' - Z(\delta, t(w_0), w_0)}{w_0} \quad (37)$$

where

$$Z(\delta, t, w) = w^2 - \exp\left(-\frac{2\zeta_{T-t-1}(\delta, t)}{\sum_{\tau=1}^t \delta^{\tau-1}}\right) \left(w - \sum_{\tau=t+1}^T \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)}\right)^2$$

Observe that $Z(\delta, t(w_0), w_0) > 0$ iff³⁷

$$\sum_{\tau=t(w_0)+1}^T \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)} > \left(1 - e^{\left(\frac{\zeta_{T-t(w_0)-1}(\delta, t(w_0))}{\sum_{\tau=1}^{t(w_0)} \delta^{\tau-1}}\right)}\right) w_0$$

For sufficiently small δ , this holds for any values of $\{r_i\}_{i=1}^{T-1}$, w_0 and $\{c_\tau\}_{\tau=t(w_0)}^T$.³⁸ Moreover, if $t(w_0)+1 > \frac{\sum_{\tau=1}^{t(w_0)} \tau \delta^\tau}{\sum_{\tau=1}^{t(w_0)} \delta^\tau}$, a sufficient condition for it to hold for all values of δ is $\sum_{\tau=t(w_0)+1}^T \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)} >$

³⁵ $\frac{\partial F(g'', l'', x)}{\partial x} > 0$ is equivalent to $(w_0 - x)^2 [2(w_0 - x) + g'' - l''] > 2(w_0 - x)(w_0 + g'' - x)(w_0 - l'' - x)$ or $(w_0 - x)(g'' - l'') < 2g'' l''$. (36) suffices for the last inequality to hold.

³⁶ For example, consider the following conditions: $\Delta_t g(t; \tilde{w}_0'') = \frac{V(t; w_0) \Delta_t g(t; w_0)}{\mathbb{E}_{\tilde{w}_0''} [V(t; \tilde{w}_0'')]}$ and $g(t+1; \tilde{w}_0'') \geq g(t+1; w_0)$. Under these, $\mathbb{E}_{\tilde{w}_0''} \Delta_t V(t; \tilde{w}_0'') > \Delta_t V(t; w_0)$ suffices for $\Delta_t [g(1; \tilde{w}_0'') \mathbb{E}_{\tilde{w}_0''} [V(1; \tilde{w}_0'')]] > \Delta_t [g(1; w_0) V(1; w_0)]$.

³⁷ Consider $Z(\delta, t, w_0)$ as a quadratic in $\sum_{\tau=t+1}^T \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)}$: $Z(\delta, t, w_0) > 0$ iff $\left(1 - \exp\left(\frac{\zeta_{T-t-1}(\delta, t)}{\sum_{\tau=1}^t \delta^{\tau-1}}\right)\right) w_0 < \sum_{\tau=t+1}^T \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)} < \left(1 + \exp\left(\frac{\zeta_{T-t-1}(\delta, t)}{\sum_{\tau=1}^t \delta^{\tau-1}}\right)\right) w_0$. By (30), the second inequality holds trivially.

³⁸ Since $\lim_{\delta \rightarrow 0^+} \log\left(\frac{\sum_{\tau=1}^t \delta^{\tau-1}}{\sum_{\tau=1}^{t+k} \delta^{\tau-1}}\right) = 0 = -\lim_{\delta \rightarrow 0^+} \left(\frac{\sum_{\tau=1}^{t+k} \delta^{\tau-1}}{\sum_{\tau=1}^t \delta^{\tau-1}}\right) \log\left(\sum_{\tau=1}^{t+k} \delta^{\tau-1}\right)$ and $\lim_{\delta \rightarrow 0^+} \frac{\sum_{\tau=t+1}^{t+k} \delta^{\tau-1} \log \delta^{\tau-1}}{\sum_{\tau=1}^t \delta^{\tau-1}} = 0^-$, we have $\lim_{\delta \rightarrow 0^+} \frac{\zeta_k(\delta, t)}{\sum_{\tau=1}^t \delta^{\tau-1}} = 0^-$. Let $k = T - t(w_0) - 1$ for the result in the text.

$$\left(T^{\frac{T}{t(w_0)+1}} - t(w_0) - 1\right) T^{-\frac{T}{t(w_0)+1}} w_0. \text{ }^{39}$$

Notice also that $Z(\delta, w_0, t)$ increases with $\sum_{\tau=t+1}^T \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)}$.⁴⁰ In other words, the lower bound for $g'' - l''$ in (37) falls with the present value of the amount of initial wealth allocated for consumption optimally under the certain prospect. Moreover, if

$$\frac{w_0 - \sum_{\tau=t(w_0)+2}^T \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)}}{w_0 - \sum_{\tau=t(w_0)+1}^T \frac{c_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)}} > \exp\left(\frac{2\zeta(\delta, t(w_0) + 1)}{\sum_{\tau=1}^{t(w_0)+1} \delta^{\tau-1}} - \frac{2\zeta(\delta, t(w_0))}{\sum_{\tau=1}^{t(w_0)} \delta^{\tau-1}}\right) \quad (38)$$

the lower bound decreases the shorter is the optimal horizon under the certain prospect (or small gambles), $t(w_0)$.⁴¹

A.2 Extension to Lotto-type Bets

Let us now consider gambles $(g^*, p; -l^*, 1-p) : p \rightarrow 0$ with again only l^* being a small stake. A minuscule probability of winning simplifies significantly the mathematical exposition of the argu-

³⁹We have

$$\begin{aligned} \frac{\partial}{\partial \delta} \left(\frac{\zeta_{T-t-1}(\delta, t)}{\sum_{\tau=1}^t \delta^{\tau-1}} \right) &= \frac{\sum_{\tau=2}^t (\tau-1) \delta^{\tau-2}}{\sum_{\tau=1}^t \delta^{\tau-1}} - \frac{\sum_{\tau=2}^T (\tau-1) \delta^{\tau-2}}{\sum_{\tau=1}^T \delta^{\tau-1}} \\ &+ \frac{\sum_{\tau=t+1}^{T-1} (\tau-1) \delta^{\tau-2}}{\sum_{\tau=1}^t \delta^{\tau-1}} - \frac{\sum_{\tau=t+1}^{T-1} \delta^{\tau-1}}{\sum_{\tau=1}^t \delta^{\tau-1}} \frac{\sum_{\tau=2}^T (\tau-1) \delta^{\tau-2}}{\sum_{\tau=1}^T \delta^{\tau-1}} \\ &+ \left(\sum_{\tau=1}^t \delta^{\tau-1} \right)^{-1} \sum_{\tau=t+1}^{T-1} (\tau-1) \delta^{\tau-2} \log \left(\frac{\delta^{\tau-1}}{\sum_{\tau=1}^T \delta^{\tau-1}} \right) \\ &- \left(\sum_{\tau=1}^t \delta^{\tau-1} \right)^{-2} \sum_{\tau=2}^t (\tau-1) \delta^{\tau-2} \sum_{\tau=t+1}^{T-1} \delta^{\tau-1} \log \left(\frac{\delta^{t-1}}{\sum_{\tau=1}^T \delta^{\tau-1}} \right) \\ &= \frac{\sum_{\tau=2}^T (\tau-1) \delta^{\tau-2}}{\sum_{\tau=1}^t \delta^{\tau-1}} - \frac{\sum_{\tau=2}^T (\tau-1) \delta^{\tau-2}}{\sum_{\tau=1}^T \delta^{\tau-1}} \left(1 + \frac{\sum_{\tau=t+1}^T \delta^{\tau-1}}{\sum_{\tau=1}^t \delta^{\tau-1}} \right) \\ &+ \left(\sum_{\tau=1}^t \delta^{\tau-1} \right)^{-2} \left(\frac{\sum_{\tau=1}^t \delta^{\tau-1} \sum_{\tau=t+1}^T (\tau-1) \delta^{\tau-2} \log \left(\frac{\delta^{\tau-1}}{\sum_{\tau=1}^T \delta^{\tau-1}} \right)}{-\sum_{\tau=2}^t (\tau-1) \delta^{\tau-2} \sum_{\tau=t+1}^T \delta^{\tau-1} \log \left(\frac{\delta^{\tau-1}}{\sum_{\tau=1}^T \delta^{\tau-1}} \right)} \right) \\ &= \left(\sum_{\tau=1}^t \delta^{\tau-1} \right)^{-2} \left(\frac{\sum_{\tau=t+1}^T (\tau-1) \delta^{\tau-2} \log \left(\frac{\delta^{\tau-1}}{\sum_{\tau=1}^T \delta^{\tau-1}} \right)}{+\sum_{\tau=2}^t \sum_{\tau=t+1}^T (t-\tau) \delta^{t+\tau-3} \log \left(\frac{\delta^{t-1}}{\sum_{\tau=1}^T \delta^{\tau-1}} \right)} \right) \end{aligned}$$

This expression is not of the same sign across all $T \in \mathbb{N} \setminus \{0\}, t \in \{1, \dots, T-1\}$. A sufficient condition for it to be negative is $t+1 > \frac{\sum_{\tau=1}^t \tau \delta^\tau}{\sum_{\tau=1}^t \delta^{\tau-1}}$. In this case, $\inf_{\delta \in (0,1)} \exp\left(\frac{\zeta_{T-t-1}(\delta, t)}{\sum_{\tau=1}^t \delta^{\tau-1}}\right) = \exp\left(\frac{\zeta_{T-t-1}(1, t)}{t+1}\right)$ where $\zeta_{T-t-1}(1, t) = (t+1) \log(t+1) - T \log T$.

⁴⁰Given $\lambda_t > 0$ and $w_0 > x_{t+1}$, $-\lambda_t (w_0 - x_{t+1})^2$ is strictly increasing in x_{t+1} . In $Z(\delta, w_0, t)$, set $\lambda_t = \exp\left(-\frac{2\zeta_{T-t-1}(\delta, t)}{\sum_{\tau=1}^t \delta^{\tau-1}}\right)$. We have $w_0 > x_{t+1}$ by (30).

⁴¹ $\Delta_t Z(\delta, w_0, t) = -\lambda_{t+1} (w_0 - x_{t+2})^2 + \lambda_t (w_0 - x_{t+1})^2 < - (w_0 - x_{t+1}) [\lambda_{t+1} (w_0 - x_{t+2}) - \lambda_t (w_0 - x_{t+1})]$ where λ_t is defined in the preceding footnote. The inequality follows from x_{t+k} being decreasing in k .

ment. In any case, it is appropriate for lotto-type bets (for such gambles, p is usually of the order 5×10^{-9} when one buys one lotto entry for $l^* = \$1$). Observe that

$$\begin{aligned} & \mathbb{E}_{\tilde{w}_0^*} [V(t; \tilde{w}_0^*)] \\ &= \sum_{\tau=1}^t \delta^{\tau-1} \left(\begin{aligned} & \log \left(w_0 + g^* - \sum_{\tau=t+1}^T c_\tau \prod_{i=1}^{\tau-1} (1+r_i)^{-1} \right)^p \\ & + \log \left(w_0 - l^* - \sum_{\tau=t+1}^T c_\tau \prod_{i=1}^{\tau-1} (1+r_i)^{-1} \right)^{1-p} \end{aligned} \right) \\ & \quad + \sum_{\tau=t+1}^T \delta^{\tau-1} \log c_\tau - \sum_{\tau=1}^t \delta^{\tau-1} \log \left(\sum_{\tau=1}^t \delta^{\tau-1} \right) + \sum_{\tau=2}^t \delta^{\tau-1} \log \left(\delta^{\tau-1} \prod_{i=1}^{\tau-1} (1+r_i) \right) \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E}_{\tilde{w}_0^*} [\Delta_t V(t; \tilde{w}_0^*)] - \Delta_t V(t; w_0) \\ &= \sum_{\tau=1}^t \delta^{\tau-1} \Delta_t \left(\begin{aligned} & \log \left(w_0 + g^* - \sum_{\tau=t+1}^T c_\tau \prod_{i=1}^{\tau-1} (1+r_i)^{-1} \right)^p \\ & + \log \left(w_0 - l^* - \sum_{\tau=t+1}^T c_\tau \prod_{i=1}^{\tau-1} (1+r_i)^{-1} \right)^{1-p} \\ & - \log \left(w_0 - \sum_{\tau=t+1}^T c_\tau \prod_{i=1}^{\tau-1} (1+r_i)^{-1} \right) \end{aligned} \right) \\ & \quad + \delta^t \log \left[\frac{\left(w_0 + g^* - \sum_{\tau=t+2}^T c_\tau \prod_{i=1}^{\tau-1} (1+r_i)^{-1} \right)^p \left(w_0 - l^* - \sum_{\tau=t+2}^T c_\tau \prod_{i=1}^{\tau-1} (1+r_i)^{-1} \right)^{1-p}}{w_0 - \sum_{\tau=t+2}^T c_\tau \prod_{i=1}^{\tau-1} (1+r_i)^{-1}} \right] \end{aligned}$$

Thus, a sufficient condition for $\mathbb{E}_{\tilde{w}_0^*} [\Delta_t V(t; \tilde{w}_0^*)] > \Delta_t V(t; w_0)$ is the following

$$\delta^t \log F(g^*, l^*, x_{t+2}) < \sum_{\tau=1}^t \delta^{\tau-1} \log \left(\frac{F(g^*, l^*, x_{t+1})}{F(g^*, l^*, x_{t+2})} \right) \quad (39)$$

where $F : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ is defined by $F(g, l, x) = \frac{(w_0 - x)^2}{(w_0 + g - x)^p (w_0 - l - x)^{1-p}}$. Observe now that, for given $p > 0$, the left-hand side of (39) tends to $-\infty$ as $g \rightarrow +\infty$. In contrast, the right-hand side has a finite lower bound.⁴² That is, for every p , there exists some large enough gain g^* so that (39) holds.

Therefore (recall Footnote 37), the lotto-type bet can induce a longer optimal horizon than that of the certain prospect, $t(\tilde{w}_0^*) = t(w_0) + k$ for some $k \geq 1$. In this case, the following claim establishes a sufficient condition for the gamble to be accepted.

Claim 6 *For a gamble $(g^*, p; -l^*, 1-p)$, suppose that it induces a longer optimal horizon than the*

⁴²We have $\frac{F(g, l, x_{t+1})}{F(g, l, x_{t+2})} = \left(\frac{w_0 - x_{t+1}}{w_0 - x_{t+2}} \right)^2 \left(\frac{w_0 + g - x_{t+2}}{w_0 + g - x_{t+1}} \right)^p \left(\frac{w_0 - l - x_{t+2}}{w_0 - l - x_{t+1}} \right)^{1-p} > \left(\frac{w_0 - x_{t+1}}{w_0 - x_{t+2}} \right)^2$, since $x_{t+1} > x_{t+2}$.

certain prospect, $t(\tilde{w}_0^*) = t(w_0) + k : 1 \leq k \leq T - t(w_0)$. If

$$\begin{aligned} & \exp\left(-\frac{2\zeta_k(\delta, t(w_0))}{\sum_{\tau=1}^{t(w_0)} \delta^{\tau-1}}\right) \left(w_0 - \sum_{\tau=t(w_0)+1}^T \frac{\underline{c}_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)} \right) \\ & \leq \left(w_0 + g^* - \sum_{\tau=t(w_0)+k+1}^T \frac{\underline{c}_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)} \right)^p \left(w_0 - l^* - \sum_{\tau=t(w_0)+k+1}^T \frac{\underline{c}_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)} \right)^{1-p} \end{aligned} \quad (40)$$

then the gamble will be accepted.

Proof. We want $\mathbb{E}_{\tilde{w}_0} [V(t(w_0) + k; \tilde{w}_0^*)] > V(t(w_0); w_0)$ for $1 \leq k \leq T - t(w_0)$. The argument is identical to that in the proof of Claim 4, since by (30) we get

$$\left(w_0 + g^* - \sum_{\tau=t+k+1}^T \frac{\underline{c}_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)} \right)^p \left(w_0 - l^* - \sum_{\tau=t+k+1}^T \frac{\underline{c}_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)} \right)^{1-p} > \sum_{\tau=t+1}^{t+k} \frac{\underline{c}_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)}$$

for any $1 \leq t \leq T - k$. ■

Given g^* , let $p \rightarrow 0$. (40) will hold if

$$\begin{aligned} & \left(e^{-\frac{2\zeta_k(\delta, t(w_0))}{\sum_{\tau=1}^{t(w_0)} \delta^{\tau-1}}} - 1 \right) \left(w_0 - \sum_{\tau=t(w_0)+k+1}^T \frac{\underline{c}_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)} \right) \\ & \leq e^{-\frac{2\zeta_k(\delta, t(w_0))}{\sum_{\tau=1}^{t(w_0)} \delta^{\tau-1}}} \sum_{\tau=t(w_0)+1}^{t(w_0)+k} \frac{\underline{c}_\tau}{\prod_{i=1}^{\tau-1} (1+r_i)} - l^* \end{aligned} \quad (41)$$

Since $\lim_{\delta \rightarrow 0^+} \frac{\zeta_k(\delta, t)}{\sum_{\tau=1}^t \delta^{\tau-1}} = 0^-$ (recall footnote 33), this inequality is valid for sufficiently low values of δ . For every g^* , there exist pairs (δ, p) , with the discount factor and winning probability both being small enough, so that (40) holds.⁴³

It is of interest to note that (41) becomes easier to satisfy the lower is the initial endowment w_0 , other things remaining unchanged.

⁴³Since a larger g^* makes (40) more likely to hold, this argument is valid in conjunction with the one given for why (39) should hold. That is, for sufficiently small δ , there do exist pairs (p, g^*) with p very small but g^* sufficiently large such that both (40) and (39) are satisfied.