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Future directions in automorphisms of surfaces, graphs, and other related topics

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Abstract. The study of Riemann surfaces and the groups which act on them is a classical area of research dating back to the latter half of the 19th century. Research in this field has wide-reaching implications in geometry and topology, algebra, combinatorics, analysis, and number theory through related topics such as the study of dessins d'enfants, mapping class groups, and graphs on surfaces. Today, this is still a rich area of research with many open questions. In this expository article we pose 78 open problems, contextualize them within the field, and discuss partial results or progress toward answering the questions, when relevant.

1. Introduction

As described in the preface, this volume focuses on the interests of a loose international collaboration of dozens of researchers. At the heart of the collaboration is a series of conferences and special sessions which have taken place over the last two decades, and where over a hundred talks have been presented by more than fifty different researchers. During this time, some of the senior researchers who laid the foundations of the field in the 60's and 70's have retired from active research, junior mathematicians who first entered the field at the start of this conference series have grown to become leaders, and new researchers have joined, and continue to join, the community. With an eye toward the future of the field, in this article we explore possible areas of future research, drawing upon both the persistent themes from the conferences, as well as ideas submitted to us when we surveyed the research community. These areas of research blend geometry and topology, algebra, combinatorics, analysis, and number theory. This broad intersection with various areas of mathematics helps to explain the longevity of this field.

The outline of this article is as follows. In Section 2 we provide some preliminary material in order to help the reader understand many of the problems in the later sections. Readers who already work in the field may comfortably skip Section 2. The remaining sections focus on open problems and context around those problems. In Section 3 we consider problems about which groups can act on surfaces

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with given genus. In Section 4 we consider parametric families of surfaces and the variation of the automorphism groups within the family. Of particular interest are the moduli space, the Teichmüller space and Hurwitz spaces. Hyperelliptic and superelliptic surfaces have the most tractable equations so that we can ask deeper questions about these surfaces. This, as well as other questions about curves, is the focus of Section 5. Section 6 covers dessins d'enfant and other graphs on surfaces, and quasiplatonic surfaces, while Section 7 discusses which surfaces support symmetries, i.e., anti-conformal, involutary isometries. Finally, Section 8 explores algorithmic and explicit methods. Many of the problems in other sections have a strong computational component and we make links to those problems in this section.

There are two caveats. First, though we have tried to include a thorough list of references, with the sheer scope of the available literature it is possible we missed some relevant references. Any such omissions are by accident. Second, to the best of our knowledge, all the posed problems throughout this article remain open. However, with the fast changing nature of mathematics, and once again with the extensive literature in the area, there may be progress on some of these problems that we are not aware of.

2. Preliminaries

In this section we introduce some preliminary material which will help the reader understand many of the open problems outlined in this article. Readers who already work in the field may skip this section.

2.1. Conformal group actions on surfaces and their construction. To understand the problems posed in this paper we need to understand conformal group actions, the construction of surfaces with actions, and their principal tools of study. Surfaces with actions may be constructed in a variety of ways such as by monodromy epimorphisms, surface kernel epimorphisms, defining equations, and tilings by fundamental domains. The principal tools of study are signatures, generating vectors, defining equations, and equivalence relations. More detail can be found in the articles [**20**] and [**25**]. We introduce construction and equivalence of actions in this section. Defining equations and tilings by fundamental domains are deferred to Sections 3.2 and 6.3, respectively.

Definition 2.1. The finite group G *acts conformally* on the Riemann surface S through a monomorphism:

$$\epsilon : G \to \mathrm{Aut}(S)$$

where $\mathrm{Aut}(S)$ denotes the full group of conformal automorphisms of S. The subgroup $\epsilon(G)$ is called the *image subgroup* or *action image*.

Focusing on the action ϵ (using generating vectors) instead of the image subgroup $\epsilon(G)$, allows us to consider the group G as the primary object. It also allows us to construct surfaces with prescribed automorphisms without having to find defining equations of the surface or worry about computing the full automorphism group. The subgroup $\epsilon(G) \subseteq \mathrm{Aut}(S)$ is determined by any of the actions $\epsilon \circ \omega$ for $\omega \in \mathrm{Aut}(G)$.

2.1.1. *Monodromy epimorphisms.* The quotient surface $S/G = T$ is a closed Riemann surface of genus τ with a unique conformal structure so that

$$\pi_G : S \to S/G = T$$

is holomorphic.

The quotient map $\pi_G : S \to T$ is ramified uniformly over a finite set $B_G = \{Q_1, \ldots, Q_t\}$ such that π_G is an unramified covering exactly over $T^\circ = T - B_G$. We say that S *lies over* (T, B_G).

Let $S^\circ = \pi_G^{-1}(T^\circ)$ so that $\pi_G : S^\circ \to T^\circ$ is an unramified covering space whose group of deck transformations, $\mathrm{Gal}(S^\circ/T^\circ) = \mathrm{Gal}(S/T)$, equals $\epsilon(G)$ restricted to S°. This covering determines a normal subgroup $\Pi_G = \pi_1(S^\circ) \lhd \pi_1(T^\circ)$ and an exact sequence $\Pi_G \hookrightarrow \pi_1(T^\circ) \twoheadrightarrow \epsilon(G)$ by mapping loops to deck transformations. Combine the last map with $\epsilon(G) \stackrel{\epsilon^{-1}}{\to} G$ to get an exact sequence

$$\Pi_\xi = \Pi_G \hookrightarrow \pi_1(T^\circ) \stackrel{\xi}{\twoheadrightarrow} G, \tag{2.1}$$

using the notation Π_ξ if we need to distinguish different maps with the same T°. We call ξ a *(regular) monodromy epimorphism.* We have left out base points to simplify the exposition, and so ξ is ambiguous up to inner automorphisms, but this is inconsequential.

REMARK 2.2. The map from epimorphisms ξ to actions ϵ is given by

$$\xi \to \epsilon = \widetilde{\xi^{-1}}, \tag{2.2}$$

where the tilde is the homomorphism $\pi_1(T^\circ) \to \pi_1(T^\circ)/\Pi_G \to \mathrm{Aut}(S)$, defined by lifting paths to deck transformations.

The fundamental group $\pi_1(T^\circ)$ has the following presentation:

$$\text{generators} \quad : \quad \{\alpha_i, \beta_i, \gamma_j, 1 \le i \le \tau, 1 \le j \le t\}, \tag{2.3}$$

$$\text{relation} \quad : \quad \prod_{i=1}^{\tau} [\alpha_i, \beta_i] \prod_{j=1}^{t} \gamma_j = 1. \tag{2.4}$$

The generating set

$$\mathcal{G} = (\alpha_1, \ldots, \alpha_\tau, \beta_1, \ldots, \beta_\tau, \gamma_1, \ldots, \gamma_t) \tag{2.5}$$

is not unique, but typically it is fixed for a discussion of all surfaces lying over (T, B_G).

Define

$$a_i = \xi(\alpha_i), b_i = \xi(\beta_i), c_j = \xi(\gamma_j). \tag{2.6}$$

Then the $(2\tau + t)$-tuple

$$\mathcal{V} = (a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t) \tag{2.7}$$

is called a *generating vector* for the action. We observe that

$$G = \langle a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t \rangle . \tag{2.8}$$

Defining

$$n_j = o(c_j), \tag{2.9}$$

we also have

$$\prod_{i=1}^{\tau}[a_i,b_i]\prod_{j=1}^{t}c_j = c_1^{n_1} = \cdots = c_t^{n_t} = 1. \tag{2.10}$$

The $(t+1)$-tuple $(\tau; n_1, \ldots, n_t)$ is called the *signature* of the action (more precisely, of the generating vector). For n-gonal actions, i.e., S/G is a sphere so that $\tau = 0$, we write $(n_1, \ldots, n_t)$. The number τ is called the *orbit genus,* namely the genus of T, and the n_j are called the *periods* of the action. The possible signatures are constrained by the Riemann Hurwitz equation

$$\frac{2g-2}{|G|} = 2\tau - 2 + \sum_{j=1}^{t}\left(1 - \frac{1}{n_j}\right). \tag{2.11}$$

Now, given a generating vector $\mathcal{V}$ satisfying equations 2.8 - 2.10 above, a regular monodromy epimorphism, as in equation 2.1, is determined by the equations 2.6. The subgroup $\Pi_\xi = \ker(\xi)$ may be used to construct a covering space S° of T°, whose compactification S is a Riemann surface with G-action, quotient $T = S/G$, and the given generating vector and signature. The genus of S is determined by the Riemann-Hurwitz equation 2.11. Thus, given the information (T, B_G) and $\mathcal{V}$, a monodromy epimorphism may be constructed. We summarize the correspondence:

$$\epsilon : G \to \mathrm{Aut}(S) \quad \longleftrightarrow \quad \Pi_\xi \hookrightarrow \pi_1(T^\circ) \overset{\xi}{\twoheadrightarrow} G \quad \longleftrightarrow \quad (T, B_G) \text{ and } \mathcal{V}. \tag{2.12}$$

2.1.2. *Surface kernel epimorphisms.* Monodromy epimorphisms allow us to construct actions using topological methods. To put geometrical structures on the surface we use uniformizing Fuchsian groups and surface-kernel epimorphisms, a homomorphic version of a monodromy epimorphism.

Consider any Fuchsian group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ with signature $(\tau; n_1, \ldots, n_t)$. The group Γ has a presentation similar to that for $\pi_1(T^\circ)$; the generators of Γ are given by 2.3, satisfying the relation 2.4, as well as the additional relations

$$\gamma_j^{n_j} = 1, \; j = 1, \ldots, t.$$

Given a generating vector $\mathcal{V}$ as in equation 2.7, we may define a map $\eta : \Gamma \to G$ using the formulas similar to those for ξ in equation 2.6. The kernel of η is a torsion free Fuchsian group, isomorphic to Π_G; we denote it by Π_η. We get an exact sequence

$$\Pi_\eta \hookrightarrow \Gamma \overset{\eta}{\twoheadrightarrow} G.$$

We set $S = \mathbb{H}/\Pi_\eta$ and the action of G is given by

$$\eta \to \epsilon = \overline{\eta^{-1}},$$

where the overbar denotes the natural quotient action of Γ/Π_η on $\mathbb{H}/\Pi_\eta$.

Starting with a quotient surface pair (T, B_G) and a generating vector $\mathcal{V}$ with signature $(\tau; n_1, \ldots, n_t)$, a Fuchsian group Γ may be constructed such that $T = \mathbb{H}/\Gamma$ and $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ is branched over B_G. The group Γ and the map π are uniquely determined up to a conjugating element in $\mathrm{PSL}(2,\mathbb{R})$. Then, using $\mathcal{V}$ to get $\eta : \Gamma \to G$, we obtain the correspondence below, analogous to 2.12:

$$\epsilon : G \to \mathrm{Aut}(S) \quad \longleftrightarrow \quad \Pi_\eta \hookrightarrow \Gamma \overset{\eta}{\twoheadrightarrow} G \quad \longleftrightarrow \quad (T, B_G) \text{ and } \mathcal{V}. \tag{2.13}$$

REMARK 2.3. The correspondence $\xi \leftrightarrow \eta$ and the resulting relation between 2.12 and 2.13 is induced by a surjective homomorphism $\pi_1(T^\circ) \to \Gamma$. In turn, this homomorphism maps the generating set $\mathcal{G}$ (equation 2.5) to a generating set for Γ. We trust that using the same notation for both generating sets will not lead to confusion. Using this identification, generating vectors may also be constructed from surface kernel epimorphisms using equations analogous to 2.6 and 2.7. In this case $o(\gamma_j) = o(c_j)$ so that Π_η is torsion free.

The pair (T, B_G) and $\mathcal{V}$ determine the same surface and action in both associations 2.12 and 2.13, as long as $\mathcal{G}$ has the interpretation above. Thus (T, B_G) and $\mathcal{V}$ and may be used to specify surfaces and actions, which we call the *quotient viewpoint*. The value of this viewpoint is that (T, B_G) captures "continuous moduli", whilst the vectors $\mathcal{V}$ are finite in number and for each (T, B_G) there are only finitely many surfaces S lying over (T, B_G). In addition, the generating vectors $\mathcal{V}$ can be enumerated, computed directly if needed, and analyzed using finite group theoretic methods.

2.2. Equivalence of actions. Later in Section 3 we will consider group actions under various types of equivalence. For a detailed discussion of equivalence relations on actions, see [20] in this proceedings. We will focus on four types of equivalences.

DEFINITION 2.4. Two actions ϵ_1, ϵ_2 of G, on possibly different surfaces S_1, S_2 of the same genus, are *topologically equivalent* if there is an intertwining, orientation preserving homeomorphism $h : S_1 \to S_2$ and an automorphism $\omega \in \mathrm{Aut}(G)$ such that

$$\epsilon_2(a) = h\epsilon_1(\omega(a))h^{-1}, \forall a \in G. \tag{2.14}$$

DEFINITION 2.5. In the definition above, if h is a conformal map, then we say that the actions are *conformally equivalent*.

Geometrically, the mapping part of the equivalence equation 2.14 translates the commutative diagram of branched covers:

$$\begin{array}{ccc} S_1 & \xrightarrow{h} & S_2 \\ \downarrow{\scriptstyle \pi_{\epsilon_1}} & & \downarrow{\scriptstyle \pi_{\epsilon_2}} \\ T_1 & \xrightarrow{\overline{h}} & T_2 \end{array}$$

where the vertical arrow are different group action quotients by G and $\overline{h}$ is the induced map on quotients. The diagram allows us to see that each equivalence class of actions can be viewed as a set (which is finite) of surfaces which are regular branched covers of a representative (T, B_G) of an equivalence class of quotient pairs.

Topological equivalence classes can be interpreted as the orbits of a *topological action* on any one of G-actions, monodromy epimorphisms, surface kernel epimorphisms, or generating vectors. Starting with G-actions on surfaces we write the action as

$$\epsilon_2 = Ad_h \circ \epsilon_1 \circ \omega^{-1},$$

where $\mathrm{Ad}_h(k) = h \circ k \circ h^{-1}$. This transfers to monodromies:

$$\xi_2 = \omega \circ \xi_1 \circ (\overline{h}_*)^{-1}, \tag{2.15}$$

according to equation 2.2. For a discussion of the action on surface-kernel epimorphisms see Section 8.2. The action on generating vectors is obtained by combining equations 2.15 and 2.6. This action on generating vectors is particularly useful for computer determination of topological equivalence classes, especially the braid operations described in Example 2.8 below.

Remark 2.6. The condition 2.15 is slightly different than the one given in equation 2.14. However, the formula in 2.14 is simpler to state and yields the same equivalence relation.

Remark 2.7. Two actions related by $\epsilon_2 = \epsilon_1 \circ \omega^{-1}$ and hence $\xi_2 = \omega \circ \xi_1$, determine the same surface S with $\mathrm{Aut}(G)$ equivalent actions. Thus we really only need to consider the action of the various $(\overline{h}_*)^{-1}$ on the $\mathrm{Aut}(G)$ classes of generating vectors to enumerate and analyze the surfaces lying over (T, B_G), with the given signature.

A major interest for the action of the various $(\overline{h}_*)^{-1}$ are the so-called braid operations described in the next example.

Example 2.8. For n-gonal actions the $\mathrm{Aut}(G)$-action (Remark 2.7) along with the *basic braid maps* $(\overline{h}_*)^{-1}$, described following follows, generate the topological action. Consider a homeomorphism which switches exactly two of the branch points $Q_j \leftrightarrow Q_{j+1}$, even though this may alter the signature. Among the possibilities, a homeomorphism $\overline{h}$ may be found such that

$$(\overline{h}_*)^{-1} : \gamma_j \to \gamma_j \gamma_{j+1} \gamma_j^{-1}, \gamma_{j+1} \to \gamma_j,$$

and all other generators fixed. The action on generating vectors is

$$c_j \to c_j c_{j+1} c_j^{-1},\ c_{j+1} \to c_j,$$

and all other elements fixed. We shall call any composition chain of the above braid maps a *braid operation.* We think of them as operators on the spaces of generating sets $\mathcal{G}$ and generating vectors $\mathcal{V}$. The name derives from the action of the braid group, which we do not describe here.

Definition 2.9. Suppose that two actions of G, on possibly different surfaces S_1, S_2 of the same genus, have the same quotient pair (T, B_G), and are defined by the monodromies ξ_1, ξ_2, with corresponding generating vectors $\mathcal{V}_1$, $\mathcal{V}_2$. Then the surfaces are called *braid equivalent* if $\mathcal{V}_1$ can be transformed to $\mathcal{V}_2$ by braid operations. The two surfaces are called *braid companions.*

Signatures are a crude invariant of actions. The G-signature, which we define next, defines an intermediate equivalence between the equivalence defined by signatures and topological equivalence. Assuming that we have a generating vector as in equation 2.7, let C_j denote the conjugacy class c_j^G and call the t-tuple $(C_1, \ldots, C_t)$ the *G-signature* or *ramification type* of the action. Just as the order of the periods does not matter for a signature, it turns out that the order of the C_j does not matter for the ramification type. For instance the braid operations simply permute the C_j.

The G-signature can also be directly computed in the image group $\epsilon(G)$. For each $\epsilon(g) \in \epsilon(G)$, fixing a given $P \in S$, let $\varepsilon(P, g)$ denote the rotation constant of $\epsilon(g)$ at P. The rotation constant $\varepsilon(P, g)$ is the multiplication on the cotangent space,

$T_P^*(S)$, induced by the transformation $\epsilon(g^{-1})$, which fixes P. We let C'_j denote the subset in $\epsilon(G)$ satisfying $\varepsilon(P,g) = \exp(-2\pi i/o(g))$ for any P lying over Q_j. It can be easily shown that $C'_j = \epsilon(C_j)$. Therefore, we also call the t-tuple $(C'_1, \ldots, C'_t)$ the ramification type. This interpretation is useful in the case of Hurwitz spaces where there need not be a specific action of G. Also, in the positive characteristic case with tame actions, the ramification type can be defined even though defining generating vectors is problematic.

DEFINITION 2.10. Two actions ϵ_1, ϵ_2 of G, on possibly different surfaces S_1, S_2 of the same genus, are called *equivalent via G-signatures* if and only if they have the same G-signatures up to permutation.

3. Automorphism groups of Riemann surfaces

The study of automorphism groups of Riemann surfaces is classical with major results dating back to Klein [54] Wiman [87], and Hurwitz [46], [47], [48]. It is also a wide-reaching area with strong links to topology, number theory and combinatorics, spanning topics like mapping class groups, Teichmüller spaces, and the Grothendieck-Teichmüller theory of dessins d'enfant.

Early methods for finding automorphism groups were ad-hoc, and focused on the geometry of the surfaces themselves. Later in the 1960's and 1970's, a more systematic study began, using uniformizing Fuchsian groups, see [61], [62]. This systematic approach laid the foundation for wide-ranging results such as classification of certain families of group actions, see for example [36], [39], [40], and [63], and analysis of Fuchsian subgroup containment and its relation to automorphism groups, see [79], [80]. More recently, significant advances in technology and the resolution of major mathematical results such as the classification of finite simple groups and the Nielsen realization problem, have provided new tools to tackle once insurmountable problems, and opened up new, fertile areas of study in the field. These advances have led to results such as the complete classification of all group actions on surfaces of low genus, see [15] and [13], they have allowed for more detailed analysis of the actions of large, complicated groups, see for example [86] and have provided insight into the structure of objects such as mapping class groups and moduli space, see for example [2] and [12].

Due to its rich history, its links to other areas, and the diversity of approaches to solving problems, there are many interesting open problems in the study of automorphism groups of Riemann surfaces. Rather than provide a long list of such questions, we shall break them up into several loose categories providing a little background material for each as well as a sampling of some of the open problems available. We refer the reader to [25] in this volume to learn more about the background and technical details behind many of these problems.

3.1. Classification results. The most traditional and direct problems are those of classification: that is, can we determine all mathematical objects that satisfy some given property. Classification results related to automorphism groups of Riemann surfaces date back to the very beginnings of work in this area and continue to be prevalent in the literature today. Many open problems still exist since, even though the technology has significantly improved over the years, many classification results still require ad-hoc style arguments to complete.

The first two problems are completely classical, but remain as big underlying problems.

Problem 1. *Fix a group G (or family of groups) and let $\sim$ be some equivalence of group actions on a surface such as those described in Section 2.2. Determine all surfaces on which G, or a member of the family, can act up to the equivalence $\sim$.*

Problem 2. *Fix a surface S or family of surfaces $\mathcal{S}$ and let $\sim$ be some equivalence of group actions on a surface. Determine all groups which act on S, or a member of $\mathcal{S}$, up to the equivalence $\sim$.*

Each of these "big" problems can be broken down to a much more granular level where there are a wealth of related open problems. For example, the action of an automorphism group on a surface S can be coarsely classified up to signature. This classification in turn is equivalent to finding subgroups of Fuchsian groups, so many open problems involve studying Fuchsian groups, their subgroups and their over groups. Such examples include the following.

Problem 3. *For a given Fuchsian group Γ, find the number of subgroups of a given index. This can be refined further by imposing additional properties on the subgroup such as the corresponding orbit genus or whether or not the subgroup is normal.*

Further related problems focus on special classes of Fuchsian groups and their quotients.

Problem 4. *Given a Fuchsian group Γ, determine whether or not it is arithmetic, meaning commensurable to a group derived from a quaternion algebra. This can be refined further by imposing additional properties on Γ such as restricting to torsion free groups, or restricting the number of generators.*

Problem 5. *For a given Fuchsian triangle group, or family of triangle groups, determine all possible quotients.*

Finally, rather than focusing on more granular classification results, there is still significant benefit in determining general tools and techniques, either computationally or theoretically, which can be used to make further progress in the area.

Problem 6. *Determine computational or theoretical methods to aide in the types of classification problems previously mentioned.*

3.2. Defining equations for surfaces and automorphisms. Another collection of classical problems related to automorphisms of Riemann surfaces centers around defining equations for surfaces, and their automorphisms. Recall that a compact Riemann surface S is a smooth, irreducible, projective curve, i.e., S is the set of common zeros in $\mathbb{P}^n(\mathbb{C})$ of a set of homogeneous polynomials, $f_1, \ldots, f_s$, of $n+1$ homogenous variables $X_0, \ldots, X_n$:

$$S = S_{f_1,\ldots,f_s} = \{X \in \mathbb{P}^n(\mathbb{C}) : f_1(X) = \cdots = f_s(X) = 0\}$$

which is smooth and irreducible. A typical scenario is to start with a single equation defining an affine plane curve

$$S_f = \left\{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\right\}.$$

To get a Riemann surface S we need to take the projective completion of S_f in $\mathbb{P}^2(\mathbb{C})$, repair the singularities through normalization, and check for irreduciblity. The resulting curve may no longer lie in $\mathbb{P}^2(\mathbb{C})$. The function f is called a *defining equation*.

There are two common types of questions centered around defining equations. The first of these focus on finding defining equations for some object, such as a compact Riemann surface, or its automorphisms, given some other information such as a generating vector for an automorphism group. The second starts with defining equations for some object, such as a surface, and instead asks to derive some other property, such as its full automorphism group. Below are two such problems, though many others exist.

PROBLEM 7. *Given a surface $S = S_f$ (or $S = S_{f_1,\ldots,f_s}$), determine explicit equations for the elements of the automorphism group* $\mathrm{Aut}(S)$*.*

PROBLEM 8. *For a surface S with a G-action constructed via a quotient surface and set of branch points (T, B_G), and a generating vector, $\mathcal{V}$, find a defining equation (or equations) for S and an explicit description of the G-action.*

Being classical in nature, partial results and related problems appear widely in the literature, see for example [69], [76], [77],[83]. The most tractable of surfaces with defining equations are cyclic n-gonal surfaces with equation:

$$y^n = f(x),$$

where f is a rational function. The map $(x, y) \to (x, uy)$, u an nth root of unity, defines a cyclic action on S with genus zero quotient, see [19] and [26] for more details. Such curves have been extensively studied, including determining defining equations given full automorphism group in the special case where $n = p$, a prime, see [90] and [91].

3.3. The genus spectrum of a group. For a given group G, we define its genus spectrum to be the integer sequence $\mathcal{I}_G$ of genera of surfaces on which G acts. In [55] it is shown that there is a sequence of integers I_G which depend only on the structure of the Sylow-subgroups of G such that $\mathcal{I}_G \subset I_G$ and the set $I_G - \mathcal{I}_G$ is finite. The smallest value of $\mathcal{I}_G$ is called the *strong symmetric genus* of G (or the *symmetric genus* if we allow orientation reversing automorphisms), and the smallest integer $n \in I_G \cap \mathcal{I}_G$ so that for $i \geq n$ the sets I_G and $\mathcal{I}_G$ are identical is called the *minimum stable genus* of G. Many open problems exist in describing the genus spectrum of a group.

PROBLEM 9. *For a given group G, determine its genus spectrum.*

PROBLEM 10. *For a given group G, determine its strong symmetric genus, its symmetric genus or its minimum stable genus.*

We can also reverse these problems instead focusing on genus. Specifically:

PROBLEM 11. *Given a sequence of positive integers, are there groups, or families of groups, that act on surfaces of each genera in this sequence?*

PROBLEM 12. *For a given positive integer n, what groups have n as their strong symmetric genus, symmetric genus or minimum stable genus?*

3.4. Relationship with subgroups of mapping class groups. Due to the resolution of the Nielsen realization problem by Kerckhoff, see [53], every finite subgroup of the mapping class group MCG_g is isomorphic to the image of a group of conformal automorphisms by sending an automorphism to its homotopy class. Moreover, two group actions are topologically equivalent if and only if the corresponding subgroups in MCG_g are conjugate. Thus the study of automorphisms of Riemann surfaces up to topological equivalence is the same as the study of conjugacy classes of finite subgroups of the mapping class group (see, e.g., [14] and [40]). This equivalency yields a number of interesting open problems.

One direction of study is to translate what is known about automorphism groups into information about subgroups of MCG_g. For example, the following, which is equivalent to Problem 1 in the case where the equivalence $\sim$ is topological equivalence:

PROBLEM 13. *For a given finite group G, enumerate the number of conjugacy classes of subgroups of MCG_g isomorphic to G.*

Outside of a small number of examples, there has been little progress on this problem, and so the following, which is a special case of Problem 6, would be extremely useful.

PROBLEM 14. *Determine techniques, either computational or theoretical, to enumerate conjugacy classes of finite subgroups of MCG_g.*

In the theme of Problem 2.19 from [34], we could ask a number of questions regarding subgroups of MCG_g with specific properties, such as:

PROBLEM 15. *For a fixed g, can we determine each group G such that $G < MCG_g$ is unique up to conjugacy (or equivalently, can we determine the G-actions on a surface S of genus g that are unique up to topological equivalence)?*

PROBLEM 16. *Can we find which groups, or families of groups, are subgroups of MCG_g for all g?*

Study of the mapping class group MCG_g and its finite subgroups is a very active area independent of its link to automorphism groups of surfaces. This motivates a second direction of study translating information we know about subgroups of MCG_g into information about automorphism groups. Such open problems include the following.

PROBLEM 17. *For a set of generators of MCG_g, each of finite order (such as those given in [12]), describe the corresponding group action of each generator on a surface of genus g. Further, what can we conclude about these actions? For example, what does it tell us about the corresponding points in the branch locus of moduli space of genus g?*

3.5. Full automorphism groups and maximal group orders. The Hurwitz bound derived in [47] shows that for a given genus g, the largest order of a finite group of orientation preserving automorphisms is $84(g-1)$, and it was shown in [60] that this bound is attained for infinitely many g. Moreover, it was shown independently in both [1] and [64] that for any g, there always exists a group G of automorphisms of order $8(g+1)$ that acts on a surface S of genus g, and for infinitely many g this is the largest possible order of a finite automorphism group.

Taken together, these two results imply that the maximal order of a group of automorphisms on a surface of genus g lies between $8(g+1)$ and $84(g-1)$ with these two extremes being attained infinitely often. These observations motivate the following problems.

PROBLEM 18. *For a given genus g, what is the largest possible order of a finite group of automorphisms on a surface of genus g?*

PROBLEM 19. *For a given positive integer n what is the sequence of integers for which n is the maximum order of a group of automorphisms on a surface of each genus in this sequence?*

Though there has been significant progress in maximal group order problems such as those outlined above, see for example [4], a complete answer remains elusive. A closely related problem, which also is connected to Problems 1 and 2, is that of the maximality of G as a group of automorphisms of a surface S. For example, in [28] and [27], and independently in [75], the problem of when a conformal action always extends to some larger group of automorphisms is considered, and in [73] the problem of when a cyclic group of prime order extends topologically is considered. Though previous work does provide explicit results in certain special cases, the more general problem still exists.

PROBLEM 20. *For a given automorphism group G, or family of groups, acting on a surface S up to some equivalence $\sim$, can we determine conditions for when G, or a member of the family, acts as a full finite group of automorphisms of S?*

3.6. Signature realization. When considering the classification of group actions up to signature, Riemann's existence theorem provides two conditions which need to be checked to determine whether or not a group acts on a surface with a given signature. The first one, satisfaction of the Riemann-Hurwitz formula 2.11, is a simple arithmetic condition which is trivial to check given a specific G, provided we know the orders of the elements of G. Signatures which satisfy this arithmetic condition are called *potential signatures* and we denote the set of all potential signatures for G by $\mathcal{P}$. The second condition, the existence of a generating vector, or surface kernel epimorphism, is much more computationally difficult and where much current research lies. Signatures which satisfy the second condition (and hence necessarily the first condition, see [31]) are called *actual signatures* for G and we use the analogous notation $\mathcal{A}$ for the set of actual signatures of G. Signatures in the set $\mathcal{P} - \mathcal{A}$ are called *non-signatures* for G. Along the same lines as the big classification problems, we have the following:

PROBLEM 21. *For a given finite group G, can we determine the set $\mathcal{P} - \mathcal{A}$, the set of all non-signatures?*

For example, in [31] necessary and sufficient conditions are given for when the set of non-signatures is finite with a consequence being that all non-abelian simple groups exhibit this property.

For an arbitrary finite group G, the sets $\mathcal{P}$ and $\mathcal{A}$ seem to be quite complicated, so an alternate avenue of study is to instead analyze the sets $\mathcal{P}_g$ and $\mathcal{A}_g$ which are defined to be the subsets of $\mathcal{P}$ and $\mathcal{A}$ respectively that satisfy the Riemann-Hurwitz formula for given genus g. Restricting to these sets, the problem becomes:

PROBLEM 22. *For a given finite group G or family of finite groups, can we determine the set $\mathcal{P}_g - \mathcal{A}_g$?*

In [10], $\lim_{g\to\infty} |\mathcal{P}_g|/|\mathcal{A}_g| = 1$, so asymptotically speaking, there are very few potential signatures which are not also actual signatures once the genus gets large enough. Moreover, in [11] in this volume these differences are explicitly described for some special classes of groups, and based on this work, there seems to be a tremendous amount of stability in these sets as the genus g grows. Therefore, a related but possibly more tractable problem is the following:

PROBLEM 23. *For a given finite group G or family of groups, can we describe the set $\mathcal{P}_g - \mathcal{A}_g$ as $g \to \infty$?*

4. Families of Riemann surfaces and their moduli

Two surfaces may have similar groups of automorphisms but not be conformally equivalent. Mimicking the dichotomy in Problems 1 and 2 we are led to two avenues of discussion regarding collections of surfaces with similar automorphism groups. First, fixing the genus, describe the variation in the automorphism group among all possible surfaces of that genus. Second, fixing the group G or a family of groups, describe the collections of surfaces with G as a group of automorphisms. There are three types of spaces that help us understand families of Riemann surfaces with automorphisms: Hurwitz spaces, the moduli space, and Teichmüller space.

4.1. Hurwitz spaces. Let us first consider Hurwitz spaces. While there are many definitions of Hurwitz spaces, we pick the following ones. Fix a group G and a positive integer t and let $\mathbf{H}(G,t)$ be the set of conformal equivalence classes of pairs (S,f) such that:

(1) S is a surface of genus g,
(2) $f : S \to T = \mathbb{P}^1(\mathbb{C})$ is regular branched covering with $\mathrm{Gal}(S/T) \simeq G$, and
(3) f is branched over t distinct points.
(4) (S_1, f_1) and (S_2, f_2) are equivalent if and only if $f_2 = f_1 \circ h$ for some isomorphism $h : S_2 \to S_1$.

We can replace the equivalence relation in (4) by

$$(S_1, f_1) \sim (S_2, f_2) \Leftrightarrow \alpha \circ f_2 = f_1 \circ h$$

for some linear fractional transformation α, and h as in (4), to construct a reduced Hurwitz space $\mathbf{H}^{red}(G,t)$. There is a fiber bundle map $\mathbf{H}(G,t) \to \mathbf{H}^{red}(G,t)$ with fiber $\mathrm{PSL}(2,\mathbb{C})$. The reduced space captures all the information of $\mathbf{H}(G,t)$.

Let $B = \left(\mathbb{P}^1(\mathbb{C})\right)^t - \Delta$, where Δ is the multi-diagonal of $\left(\mathbb{P}^1(\mathbb{C})\right)^t$ consisting of tuples with at least two equal entries. The space B consists of ordered t-tuples of distinct points of elements of $\mathbb{P}^1(\mathbb{C})$. Using covering space techniques, $\mathbf{H}^{red}(G,t)$ can be given a complex manifold structure as a possibly disconnected covering space

$$\mathbf{H}^{red}(G,t) \to B, \tag{4.1}$$

and by composition with $\mathbf{H}(G,t) \to \mathbf{H}^{red}(G,t)$ a fiber bundle

$$\mathbf{H}(G,t) \to B. \tag{4.2}$$

The complex dimensions of $\mathbf{H}(G,t)$ and $\mathbf{H}^{red}(G,t)$ are t and $t-3$, respectively. A cover of the space $\mathbf{H}^{red}(G,t)$ maps in a finite-to-one fashion into the moduli space, however $\mathbf{H}(G,t)$ can be easier to work with. Here we present some problems on Hurwitz spaces.

PROBLEM 24. *Determine the components of* $\mathbf{H}(G,t)$ *and relate them to braid classes of* $\mathrm{Aut}(G)$ *classes of generating vectors. There is a similar question for* $\mathbf{H}^{red}(G,t)$. *See Problem 78.*

PROBLEM 25. *Describe the topology of* $\mathbf{H}(G,t)$ *and* $\mathbf{H}^{red}(G,t)$ *as completely as possible, using the maps 4.1 and 4.2.*

PROBLEM 26. *Describe the map from (the cover of)* $\mathbf{H}^{red}(G,t)$ *to its image in moduli space as completely as possible.*

The paper [**23**] in this volume uses the spaces $\mathbf{H}^{red}(G,4)$ for determining one-dimensional equisymmetric strata in moduli space.

PROBLEM 27. *For cyclic and abelian groups,* $\mathbf{H}(G,t)$ *and* $\mathbf{H}^{red}(G,t)$ *have numerous components. For more complex groups, find* $\mathbf{H}^{red}(G,t)$ *and* $\mathbf{H}^{red}(G,t)$ *with large numbers of components.*

PROBLEM 28. *Do the different components of* $\mathbf{H}(G,t)$ *have similar topology? Answer the same question for* $\mathbf{H}^{red}(G,t)$.

4.2. Moduli and Teichmüller spaces. As a set, the *moduli space* $\mathcal{M}_g$, is the set of closed Riemann surfaces of genus g, quotiented by the relation of conformal (bi-holomorphic) equivalence. The space $\mathcal{M}_g$ has the structure of a quasi-projective, complex variety of complex dimension $3g-3$, with a complicated singularity set, informally called the *branch locus* (at least for $g \geq 4$). The moduli space may be compactified to a complex projective variety $\overline{\mathcal{M}}_g$ by adding a boundary $\partial\mathcal{M}_g$ consisting of stable nodal curves which are limits of surfaces of genus g.

A marked Riemann surface of genus g is a pair (S,f) where S is a Riemann surface and $f : S_0 \to S$ is an orientation preserving homeomorphism (up to isotopy) from a reference surface S_0. As a set, the *Teichmüller space*, $\mathcal{T}_g$, is the set of marked Riemann surfaces of genus g, quotiented by the relation of conformal equivalence. The space $\mathcal{T}_g$ is may be realized as an open set in $\mathbb{C}^{3g-3}$, diffeomorphic to an open ball in $\mathbb{R}^{6g-6}$.

The mapping class group MCG_g acts on $\mathcal{T}_g$ by changing the marking: $(S,f) \to (S, f\circ\phi^{-1})$ for $\phi \in MCG_g$. This action is holomorphic and discontinuous, so there is complex orbifold quotient, $\mathcal{T}_g/MCG_g$, that is analytically equivalent to the moduli space $\mathcal{M}_g$. The map $\mathcal{T}_g \to \mathcal{M}_g$ given by $(f,S) \to S$ is a realization of the quotient map. The stabilizers of MCG_g action are the conformal automorphism groups:

$$\left\{\phi \in MCG_g : (S,f) \backsim (S, f\circ\phi^{-1})\right\} \simeq \mathrm{Aut}(S).$$

The branch locus $\mathcal{B}_g \subseteq \mathcal{M}_g$ is defined as the image, under $\mathcal{T}_g \to \mathcal{M}_g$ of the points with non-trivial stabilizers, i.e., the set of surfaces with non-trivial automorphisms (at least for $g \geq 3$). (Various subvarieties of the branch locus are associated to conjugacy classes of finite subgroups of the mapping class group MCG_g. Given $f : S_0 \to S$, a finite subgroup of MCG_g is determined by pulling back $\mathrm{Aut}(S)$ through f. As we vary $\phi \in MCG_g$, a conjugacy class $\Sigma(S)$, of finite subgroups, called the *symmetry type of* S, is determined by pullback through $f\circ\phi^{-1}$. Two surfaces S_1 and S_2 are equisymmetric if they have the same symmetry type: $\Sigma(S_1) = \Sigma(S_2)$. Given a finite subgroup $F < MCG_g$, then its conjugacy class (F) determines the (F)-*equisymmetric stratum* of $\mathcal{M}_g$:

$$\mathring{\mathcal{M}}_g^{(F)} = \left\{S \in \mathcal{M}_g : \Sigma(S) = (F)\right\}.$$

A related set, "stratum closure", is the image in $\mathcal{M}_g$ of the fixed point set of F on $\mathcal{T}_g$:

$$\mathcal{M}_g^{(F)} = \{S \in \mathcal{M}_g : (F) \leq \Sigma(S)\}\,.$$

If $\overset{\circ}{\mathcal{M}}{}_g^{(F)}$ is non-empty then $\mathcal{M}_g^{(F)}$ is its closure. The complex dimension of $\mathcal{M}_g^{(F)}$ is called the Teichmüller dimension of F.

For more background on $\mathcal{M}_g$, $\mathcal{T}_g$, and $\mathcal{B}_g$ see [**40**], [**14**] and [**18**]. The following problems are posed for moduli and Teichmüller space.

Problem 29. *For low genus, determine all the equisymmetric strata and determine which strata lie in the closure of a given stratum. Problem 2 with topological equivalence is relevant.*

Problem 30. *Lift the branch locus to Teichmüller space: $\widetilde{\mathcal{B}_g} \subset \mathcal{T}_g$. This set is composed of a locally finite system of fixed point subsets for all conjugacy classes of finite subgroups of the mapping class group. Each such fixed point subset is isomorphic to the Teichmüller space of a Fuchsian group. Describe as completely as possible the local and global properties of $\widetilde{\mathcal{B}_g}$.*

Problem 31. *For n-gonal surfaces S with G-action, describe the corresponding equisymmetric stratum $\mathcal{S}$ and its closure $\overline{\mathcal{S}}$ in $\mathcal{M}_g$ as completely as possible. This includes solving Problem 29. Problem 26 should also be used in solving this problem.*

Problem 32. *Let $\mathcal{S}$ be the equisymmetric stratum for an n-gonal G-action. Describe the limiting surfaces in $\overline{\mathcal{S}} \cap \partial\mathcal{M}_g$ and the G-action upon them. One might need to solve Problem 33.*

For our next problem consider the following family of cyclic n-gonal curves.

$$S_\lambda : y^n = x^{n_1}(x-1)^{n_2}(x+1)^{n_3}(x-\lambda)^{n_4},$$

where S_λ is projectively completed and desingularized. Assume that the exponents satisfy relations given in [**19**] to force the S_λ to be irreducible and unramified at ∞. The canonical map $S_\lambda \to T : (x, y) \to x$ is branched over $0, -1, 1$, and λ and, for all but finitely many λ, the surfaces are equisymmetric. However, in the moduli space as λ approaches $0, -1, 1$ the limiting surface in $\partial\mathcal{M}_g$ may not be S_0, S_{-1} or S_1.

Problem 33. *Let S_λ be a holomorphic family of surfaces in a neighborhood $D \subset \mathbb{C}$ of λ_0. Assume that the surfaces $S_\lambda, \lambda \in D - \lambda_0$ have a holomorphically varying G-action. Assume that S_λ goes to the boundary of moduli space as $\lambda \to \lambda_0$. Per the Stable Reduction Theorem, determine a limiting stable nodal surface S_{λ_0}, determine its structure, and the action of G upon it.*

5. Curves

As compact Riemann surfaces are in natural bijection with smooth, irreducible projective curves, studying Riemann surfaces provides insight into questions about curves.

5.1. Extending results from hyperelliptic and superelliptic curves. Recall that a curve S is hyperelliptic if it admits a cyclic 2-gonal automorphism which is necessarily central in $\mathrm{Aut}(S)$. Hyperelliptic curves, as natural extensions of elliptic curves, have been deeply studied and are generally quite well understood.

A further generalization of elliptic curves is the family of superelliptic curves. One common definition for a curve S to be *superelliptic* (and there is some disagreement in the literature as to the exact definition) is if it admits a cyclic n-gonal group of automorphisms which is central in $\mathrm{Aut}(S)$. As seen in Section 3.2, such curves are defined by an affine model $y^n = f(x)$ where $n \geq 2$ and $f(x)$ is a polynomial of degree at least 3. Note that hyperelliptic curves comprise the special case when $n = 2$. Malmendier and Shaska [**65**] give a thorough survey of the state of research into superelliptic curves, as well as provide 12 open problems which we will not reproduce here.

A natural next area to explore is other families of curves which are generalizations of superelliptic curves, and ask which properties that superelliptic curves exhibit extend to these other families. The most closely related curves are the cyclic n-gonal curves.

Problem 34. *Which results about superelliptic curves generalize to cyclic n-gonal curves for which the group generated by the n-gonal morphism is normal? Which results generalize to all cyclic n-gonal curves?*

There is a wide amount of research in the literature on cyclic n-gonal curves, though a starting point for this problem would be the papers [**26**] and [**56**]. The next natural family of surfaces to consider are abelian covers of the projective line.

Problem 35. *Which results about superelliptic curves generalize to elementary abelian covers of the projective line? Which results generalize to all abelian covers?*

One way to develop abelian covers is to build them up via a fiber product construction which would be a good place to start exploring Problem 35. See [**43**] in this proceedings for more information, and some general results on fiber products of connected Riemann surfaces.

Other problems of interest in the study of superelliptic curves revolve around defining equations and the fields over which such equations are defined. Formally, a *field of definition* of a curve S is a field $K \subseteq \overline{\mathbb{Q}}$ so that S may be defined as the zero locus of polynomials in $K[x]$. The intersection of all the fields of definition of a curve is called the *field of moduli* of the curve. Clearly the field of moduli is a subfield of any field of definition, but the moduli field may or may not be a field of definition itself. In general, determining if the field of moduli is a field of definition is difficult, yet also a question of significant interest. This motivates the following problems.

Problem 36. *In a similar vein to Problem 8, for a given curve, can we find defining equations, and if so, what are the minimal defining equations (those of minimal discriminant)?*

Problem 37. *For a given curve S, what are its fields of definition, and what is its field of moduli? Is the field of moduli also a field of definition?*

There are well-known examples of curves where the field of moduli is not a field of definition, see, for example [**42**], [**45**], and [**57**]. On the other hand, an affirmative answer to this question is known for two different families of curves. First, if S has a trivial automorphism group, then by Weil's Galois descent theorem [**85**] it is always possible to write defining equations for S over the field of moduli. Second, from [**88**] and [**89**], the field of moduli is a field of definition for quasiplatonic curves (see

Section 6.2 for more details on quasiplatonic curves). Since quasiplatonic curves often have large automorphism groups, it follows that Problem 37, in some respects, holds true for two extremes: curves with no automorphisms, and curves with a large number of automorphisms. Thus, a meaningful analysis of this problem requires looking at curves which admit non-trivial automorphisms, but not a significant number of them.

Since their automorphism groups are well understood, there has been significant progress in answering Problem 37 in the case of superelliptic curves. For example, in [7] it is shown that every superelliptic curve is defined over at most a quadratic extension of its field of moduli, and in [44] the authors determine for which superelliptic curves with extra automorphisms the field of moduli is a field of definition. This leads to the question of whether similar techniques used for the family of superelliptic curves can be modified or adapted to other families of curves. The most obvious case would be in the following problem.

PROBLEM 38. *For each member of the family of cyclic n-gonal surfaces, can we determine its field of moduli and is its field of moduli the same as a field of definition?*

As mentioned above, fiber products provide a way to build up abelian covers. We can ask the same questions of these constructions.

PROBLEM 39. *For a fiber product what is the field of moduli? Is it the same as a field of definition? What can the fiber product construction tell us about fields of definition and the field of moduli for abelian covers of the projective line?*

5.2. Jacobian varieties. The natural map from a curve to its Jacobian variety connects the automorphism group of the curve to endomorphisms of the Jacobian variety which leads to many interesting questions. For example, the automorphism group of a curve may be used to decompose its Jacobian variety (see for example [30] or [71]). Ekedahl and Serre [33] posed several questions about curves with completely decomposable Jacobians, i.e., those Jacobians which are isogenous to a product of elliptic curves. They ask

PROBLEM 40. *Are there curves of arbitrarily large genus with completely decomposable Jacobians? Is there a curve of every genus with a completely decomposable Jacobian?*

There has been some limited work done to try to answer this question [70], [92], [72]. There are also many interesting questions about the factors in the decompositions. Of particular interest is whether the factors of a given Jacobian variety have complex multiplication. For elliptic curves, this would mean the endomorphism ring of the curve is larger than $\mathbb{Z}$. A dimension d abelian variety A is said to be of CM-type if there is a commutative subring of dimension $2d$ over $\mathbb{Q}$ in $\mathrm{End}(A) \otimes \mathbb{Q}$. The special case of elliptic curves corresponds in both definitions.

PROBLEM 41. *Given a curve S with a decomposable Jacobian variety, which factors of the Jacobian have complex multiplication and, when they do, what are the CM-fields?*

We also note that Moonen and Oort [68] study Jacobian varieties of curves of genus g via the map from the moduli space $\mathcal{M}_g$ to the moduli space of principally polarized abelian varieties, $\mathcal{A}_g$. Their paper includes a series of open questions about Jacobian varieties, and the corresponding Torelli locus, which is the closure (under the Zariski topology) of the image of the injective map from $\mathcal{M}_g$ to $\mathcal{A}_g$.

6. Graphs, dessins d'enfant and quasiplatonic surfaces

Dating back centuries to the study of the Platonic solids, graphs and surfaces have been analyzed synonymously with insight from each providing valuable information about the other. Over the last few decades, the close link between surfaces and their graphs have resulted in the development of discrete versions of the classical theory of Riemann surfaces with graphs playing the role of Riemann surfaces and morphisms between graphs playing the role of branched coverings. Within these discrete versions, many of the fundamental ideas, such as the genus or Jacobian of a surface, and many of the major results, such as the Riemann-Hurwitz formula and Riemann-Roch, naturally adapt to their discrete counterparts, see for example [66] in this volume. This translation between the two fields naturally gives rise to new directions of study in both fields. Specifically, many open problems in Riemann surfaces translate into problems about the corresponding graphs and vice versa. In this section, we outline some of the open problems in the theory of graphs as related to Riemann surfaces. Since much of the focus on graphs in our series of special sessions is on the theory of dessins d'enfants and quasiplatonic surfaces, we primarily emphasize these topics in our discussion. However, we do provide some further directions of study at the end of this section.

6.1. Dessins d'enfants. We start with a brief summary of the main definitions, terminology and results in the theory of dessins d'enfants. For conciseness, we provide a loose general overview avoiding the more technical details, instead deferring a more detailed analysis to the widely available literature in the area such as [52] or [37].

A dessin d'enfant $\mathcal{D}$ is a bipartite graph embedded in some closed orientable surface S where its vertices are alternately colored black and white. A hypermap $\mathcal{H}$ is an ordered pair of permutations x and y generating a finite group G that acts transitively on a set $\mathcal{S}$. Given a dessin $\mathcal{D}$, we can create a hypermap $\mathcal{H}$ as follows: we let $\mathcal{S}$ be the set of edges of $\mathcal{D}$ and let x and y be the cycles corresponding to rotation of edges around the black and white vertices of the dessin, respectively. Note also that this process can be reversed – that is, given a hypermap $\mathcal{H}$, we can also construct a dessin on some compact oriented surface. This provides a natural correspondence between hypermaps and dessins.

Next, for a given hypermap $\mathcal{H}$, since G acts transitively on a set $\mathcal{S}$, it is equivalent to a permutation representation of the free group $\mathcal{F}_2 \to G$ on two generators acting on the cosets of a subgroup of finite index. Likewise, letting, $z = (xy)^{-1}$, since $xyz = 1$, it is also equivalent to the action of some Fuchsian triangle group Δ whose periods are divisible by the orders of x, y and z on the cosets of some finite index subgroup Γ. This gives rise to a pair (S, β), called a Belyi pair, where $S = \mathbb{H}/\Gamma$ is a compact Riemann surface and $\beta\colon \mathbb{H}/\Gamma \to \mathbb{H}/\Delta$ is a meromorphic function branched over three points where, as usual, $\mathbb{H}$ denotes the upper half plane. We call β a Belyi function and S a Belyi surface. Moreover, by Theorem 3.10 of [52], given a Belyi pair, (S, β) we get a corresponding pair of Fuchsian groups Δ and Γ where Δ is a triangle group whose periods are divisible by the branching orders of the meromorphic function, and Γ is a finite index subgroup, and hence a corresponding hypermap $\mathcal{H}$ given by the action of Δ on the cosets of Γ. Thus we get a correspondence between hypermaps, Belyi pairs and conjugacy classes of subgroups of finite index in the free group on two generators.

Finally, by Belyi's theorem, [5], a compact Riemann surface S is defined as a projective algebraic curve over the field $\bar{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$, if and only if there is a Belyi function on S setting up an equivalence between algebraic curves defined over $\bar{\mathbb{Q}}$ and hypermaps. Thus, in sum, the theory of dessins d'enfants, or Belyi Theory as introduced by Grothendieck in his Esquisse d'un Programme, [38], establishes equivalences between the following:

(1) complex projective algebraic curves defined over number fields;
(2) Belyi pairs;
(3) dessins;
(4) Riemann surfaces uniformised by subgroups of finite index in triangle groups;
(5) compact oriented hypermaps;
(6) pairs of permutations generating transitive finite permutation groups;
(7) conjugacy classes of subgroups of finite index in the free group of rank 2.

In some of these cases, the equivalences are fairly explicit, so that in specific examples one can translate data from one context to another, e.g., translating from pairs of permutations to conjugacy classes of subgroups of finite index in the free group of rank 2 using permutation representations. In other cases, however, this translation process is much more challenging. For example, finding an explicit model for a Belyi curve, given combinatorial data, is very difficult. This leads to the following question.

Problem 42. *Determine explicit translations between these equivalences.*

Through the correspondence with algebraic curves defined over $\bar{\mathbb{Q}}$ there is a natural action of the absolute Galois group on the objects above induced by its action on the coefficients of the polynomials and rational functions defining Belyi pairs. Determining the orbits of this group requires a good supply of Galois invariants. Some, such as genus, type, and automorphism group are simple but rather crude, in the sense that inequivalent objects (i.e., those in different orbits) can often share these properties. Therefore, an extremely useful problem is the following:

Problem 43. *Determine further invariants which are simple to apply but more discriminating than those that already exist.*

Related to this problem are the following.

Problem 44. *Given any finite set of Galois invariants (for the action of the absolute Galois group on dessins), are there always two dessins which are not Galois equivalent, but which share the given invariants?*

Problem 45. *Given two dessins, is there always a finite set of Galois invariants (they may depend on the chosen dessins) which determines if they are Galois equivalent?*

With much of this work still in its relative infancy, little is known about many of these different objects and the relations between them, even for relatively "small" examples (small here being intentionally ambiguous with various meanings such as low genus of the Riemann surface, small group order or highly symmetric dessins, see for example [41] in this volume). Many conjectures are often formulated based on specific examples, so the following would be useful:

Problem 46. *Build a database of examples of the objects listed above.*

A database containing several of the objects listed above of low degree is in progress. A description of the database and the mathematics used to create it is in [69] and the data so far may be found at [58].

6.2. Quasiplatonic surfaces. The most symmetric of dessins are the regular dessins, or those for which there exists a corresponding Belyi pair (S, β) where β is a regular covering of the projective line. In this case, we call S a quasiplatonic surface and we call the covering group G a quasiplatonic group. By the described correspondence, S is uniformized by a normal surface subgroup of a triangle Fuchsian group Δ, and G is a quotient of Δ with torsion free kernel. Quasiplatonic surfaces have applications in number theory, group theory and graph theory through the theory of dessins. However, they also provide geometric and algebraic information about Riemann surfaces and their automorphism groups. For example, quasiplatonic surfaces have relatively large automorphism groups and so their study is intimately related to Problem 18 and the other related problems. As another example, such surfaces define unique points in the moduli space $\mathcal{M}_g$ in genus g and typically lie at the intersection of many different strata in the branch locus of moduli space, so their existence can help describe these different strata.

The first and most obvious problems are those of existence and are closely tied to the classification problems in Section 3. It is well known that for every genus g there exists a quasiplatonic surface S uniformized by a triangle group with signature $(0; 2, 4, 2g+2)$, see [1] and [64]. Further infinite families of quasiplatonic surfaces have been constructed for various sequences of genera, see for example [60]. There is still no clear picture however of which groups or families of groups regularly occur as quasiplatonic groups, and which genera they appear in. These observations motivate a number of different problems.

First, a group G can be realized as a quasiplatonic group if and only if it is generated by a pair of elements. Though it is straightforward to check this condition for a specific group, it doesn't give us a wider picture of which groups, or families of groups, can appear as quasiplatonic groups. This motivates the following.

Problem 47. *Determine alternative necessary and sufficient conditions for when a group can be realized as a quasiplatonic group.*

A more granular approach is to focus on conditions for when members of a specific family are quasiplatonic. For example, an abelian group G is quasiplatonic if and only if it has 2 or fewer invariant factors.

Problem 48. *For a given family of groups, determine necessary and sufficient conditions for when a member of this family can be realized as a quasiplatonic group.*

With a greater understanding of which specific groups, or families of groups, can be realized as quasiplatonic surfaces, other more specific questions arise such as the following.

Problem 49. *For a given group G which can be realized as a quasiplatonic group, determine the corresponding signatures with which it acts as a quasiplatonic group, and the genera of surfaces on which it acts.*

Further directions could focus instead on the number of, or the growth rate of, quasiplatonic surfaces. In [**74**], for example, it is shown that the number of isomorphism classes of quasiplatonic Riemann surfaces of genus less than or equal to g has a growth of type $g^{\log g}$, but not much is known about how these surfaces are distributed over genera. In [**6**], enumeration formulas are developed for counting the number of distinct isomorphism classes of quasiplatonic cyclic groups, and these formulas are used in [**29**] to develop closed enumeration formulas for the total number of quasiplatonic cyclic group actions for a specific cyclic group, but outside of cyclic groups, similar enumeration formulas do not exist. Problems along these lines include the following.

Problem 50. *How many topologically distinct quasiplatonic actions are there of a finite group G on a given surfaces of genus g? For that same group, how many total actions are there over all genera?*

Problem 51. *Are there upper or lower bounds for the number of quasiplatonic surfaces of a fixed genus g?*

6.3. Building surfaces and actions from a tiling. Up until this point, the construction of most actions of a group G on a surface S of genus g have been defined either topologically using monodromy epimorphisms or geometrically using surface kernel epimorphisms. We shall see in this section how to construct S and G using a tiling, and we provide a number of related open problems regarding this construction.

Using notation from Section 2, given a specific pair (T, B_G), where T is a quotient surface and B_G a branching set, the surfaces lying over (T, B_G) are finite in number and can all be constructed as a tiling of identical "puzzle pieces", using the G-action as a recipe for putting the pieces together. Specifically, let $\mathcal{E}$ be an embedded graph in T such that B_G is contained in the vertex set of $\mathcal{E}$ and $D^\circ = T - \mathcal{E}$ is an open disc. Let e and v denote the number of edges and arcs of $\mathcal{E}$. We have $e = v + 2\tau - 1$, by an Euler characteristic argument (τ = genus of T).

We can lift the triple $(T, \mathcal{E}, B_G)$ to a tiling of $\mathbb{H}$ which projects to a tiling on S such that G simply transitively permutes the tiles (puzzle pieces). The pair (T, B_G) and the signature, determine the uniformizing group Γ and the map $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ up to conjugation in $\mathrm{PSL}(2, \mathbb{R})$. Since B_G is contained in $\mathcal{E}$, π is a diffeomorphism of each connected component of $\pi^{-1}(D^\circ)$ onto D°. Therefore, the inverse image $\pi^{-1}(D^\circ)$ is a disjoint collection of open curvilinear polygons in $\mathbb{H}$, $\left\{\gamma\widetilde{D^\circ} : \gamma \in \Gamma\right\}$, where $\widetilde{D^\circ}$ is a distinguished component of $\pi^{-1}(D^\circ)$. Each $\gamma\widetilde{D^\circ}$ has $2t-2$ sides and $2t-2$ vertices. The closure $\overline{\gamma\widetilde{D^\circ}}$ of each open $\gamma\widetilde{D^\circ}$ is a closed fundamental domain for Γ. Now pick a specific D_0 among the $\left\{\gamma\overline{\widetilde{D^\circ}} : \gamma \in \Gamma\right\}$. The sides of D_0 are paired by certain elements of Γ and under this pairing we can construct the triple $(T, \mathcal{E}, B_G)$. To construct S, we start with $|G|$ copies of D_0 labelled by the elements of G, written as $\{(D_0, g) : g \in G\}$. Applying the epimorphism $\eta : \Gamma \to G$ to the side pairing elements we determine how to glue the tiles, i.e., the $|G|$ elements of $\{(D_0, g) : g \in G\}$, together. Each $\mathrm{Aut}(G)$-class of epimorphisms $\Pi_\eta \hookrightarrow \Gamma \overset{\eta}{\twoheadrightarrow} G$ determines a distinct recipe for putting together the puzzle pieces.

Though there are many different avenues of research that could be pursued related to this construction, we are mainly interested in the case where T is the Riemann sphere and $\mathcal{E}$ is a tree whose vertex set consists of the t points in B_G, and has $t-1$ "nice" arcs, e.g., line segments. Of special interest is the case where the arcs in $\mathcal{E}$ form a segment of the real line, as this allows for the possibility of symmetries of S, see Section 7.

For the following, assume that T has genus zero, and that (T, B_G) and the generating set $\mathcal{G}$ for either Γ or $\pi_1(T^\circ)$ are given. We also assume that the tree $\mathcal{E}$ has vertex set B_G exactly.

Problem 52. *Given $(T, \mathcal{E}, B_G)$ and $\mathcal{G}$, determine the labelling of the vertices and sides of the fundamental region D_0, i.e., the quotient map $\partial D_0 \to \mathcal{E}$, and the side pairing transformations. See also Problem 57.*

Problem 53. *What conditions on $(T, \mathcal{E}, B_G)$ ensure that the sides of the fundamental region D_0 are geodesic arcs in $\mathbb{H}$ and not just smooth arcs?*

Problem 54. *What conditions on $(T, \mathcal{E}, B_G)$ ensure that the fundamental region D_0 is convex?*

Now we add a generating vector $\mathcal{V}$ into the mix and ask questions about S.

Problem 55. *Assume that Problem 52 is solved. Determine an algorithm and/or conditions on $(T, \mathcal{E}, B_G)$ that constructs a "nice" fundamental region for Π_η and hence a construction of S as a "nice" region with boundary identifications. Describe the boundary and the side pairing map. Specifying "nice" is part of the problem. A good starting point is symmetric quasiplatonic surfaces and n-gonal actions with a small number of branch points.*

Problem 56. *Assume that Problem 55 is solved. Determine explicitly the linear fractional transformations that determine the side pairings. These transformations generate the subgroup Π_η.*

The paper [16] can be of use in solving the preceding problem for 4 branch points.

For our next two problems we construct a Cayley graph on S. Pick an interior point $x_0 \in D_0$ and construct $2t-2$ smooth arcs, intersecting only at x_0, to interior points on the sides of D_0, meeting the sides at a right angle. The interior points must be selected so that they match under the side pairing transformations. Now translate the arcs by Γ. The resulting graph is an infinite, smooth Cayley graph for Γ with respect to the generating set of side-pairing transformations. The graph projects to a Cayley graph for G on S. The Cayley graph is dual to the graph determined by the tiling on S.

Problem 57. *Determine an algorithm for constructing the Cayley graph for the G-action as described above. This problem is related to Problem 52.*

Problem 58. *Construct a Cayley graph for a G-action as described above. Can any useful geometric information be extracted from this Cayley graph? In particular, can properties of this Cayley graph be used to distinguish surfaces and actions derived from distinct $\mathrm{Aut}(G)$ classes of generating vectors, but the same quotient pair (T, B_G)?*

For our final two problems we consider Petrie paths along the edges of the tiling on S. Briefly, a Petrie path on S is a zig-zag path along the edges of the tiling, such that no three consecutive edges lie on the same polygon. See [82], for example, for background on Petrie paths. A Petrie path is an important invariant of regular maps.

PROBLEM 59. *Determine an algorithm for constructing and describing the Petrie path for the G-regular tiling constructed above.*

PROBLEM 60. *As in Problem 58 determine if any useful geometric information can be extracted from the Petrie path, including distinguishing different surfaces arising form different generating vectors.*

6.4. Further directions. Though most of the focus on graphs in both this volume and the series of conferences we have organized is centered around dessins d'enfants, there are numerous other topics in graphs which are directly related to Riemann surfaces and their automorphism groups. One possible direction is the study of groups acting harmonically on a graph. An automorphism group acts *harmonically (or semi-regularly)* on a graph if it acts freely on the set of directed edges of the graph. Such an action is a natural discrete generalization of the group of conformal automorphism on a Riemann surface, and from this point of view, discrete versions of many well-known theorems are obtained, such as Hurwitz's classical upper bound for the maximal size of an automorphism group, [**46**], or generalizations of the Accola–Maclachlan surfaces found in [**1**] and [**64**]. Such results estimate the size of a group acting harmonically on a graph of given homological genus with some additional restrictions such as being cyclic, having an invariant set of prescribed order, or specifying the number of fixed points. These results motivate the following.

PROBLEM 61. *Find sharp upper and lower bounds for the size of abelian, nilpotent or other classes of groups acting harmonically on a graph of given genus, where the genus is the Betti number of the graph.*

Another possible direction is the study of the Jacobian for a graph which can be considered as a discrete counterpart of a notion of the Jacobian for a Riemann surface. The Jacobian for a graph is also known as a sandpile group, critical group, dollar group, or Picard group of a graph. The Jacobian group is isomorphic to the torsion part of the co-kernel of the Laplacian of a graph and is an important algebraic invariant of a finite graph. In particular, its order coincides with the number of spanning trees in the graph. Of particular interest are families of cubic graphs built up from trees of the shape I, Y, or H. These graphs are summarized in [**8**, pg. 147].

PROBLEM 62. *Find the structure of the Jacobian group for circulant graphs and their generalizations to I-graphs, Y-graphs and H-graphs.*

There are some known results about Jacobians of I-graphs [**67**]. For a generalization to circulant foliation, see [**49**].

7. Symmetries of surfaces

A *symmetry* ϕ of a surface S is an anti-conformal, involutary (order 2), automorphism ϕ of the surface, and a surface with a symmetry is called a *symmetric surface*. Symmetries of surfaces arise naturally as follows. If S has defining equations with real coefficients then complex conjugation induces a symmetry on S. Symmetries of S may also arise from reflections in the edges of kaleidoscopic polygons, i.e., S is a "hyperbolic kaleidoscope". See [**16**], [**24**], and [**17**] for background on kaleidoscopic polygons and symmetries of surfaces.

For a symmetry ϕ of a symmetric surface S, the *fixed point subset*, $\mathcal{M}_\phi = \{x \in S : \phi x = x\}$, if non-empty, consists of a finite number of disjoint circles called *ovals*. Motivated by the construction of symmetries from kaleidoscopes, the fixed point subset $\mathcal{M}_\phi$, is called the *mirror* of the symmetry. For symmetries arising as complex conjugations, the fixed point subset equals the set of real points of the curve.

The complement of the fixed point set $S - \mathcal{M}_\phi$ consists of one or two components, depending on whether the Klein surface, $S/\langle\phi\rangle$, possibly with boundary, is non-orientable or orientable. A symmetry (and hence its mirror) is said to be *separating* or *orientable* if $S - \mathcal{M}_\phi$ has two components, otherwise it is called *non-separating* or *non-orientable*. If k is the number of ovals of the mirror $\mathcal{M}_\phi$, then the *species* of ϕ is said to be k if ϕ is separating and $-k$ if ϕ is non-separating. A symmetry with empty mirror has species 0. The *symmetry type* of a surface is the list of species of all conjugacy classes of symmetries occurring in $\mathrm{Aut}^*(S) = \langle \phi, \mathrm{Aut}(S)\rangle$.

For a symmetric surface the determination of the symmetry type requires answers to three questions, which we formulate as the next problem.

Problem 63. *For any symmetric surface S answer the following:*
(1) *Classify all the symmetries of S.*
(2) *For each symmetry, compute the number of ovals in the mirror.*
(3) *For each symmetry, determine whether it is separating or not.*

7.1. Symmetries of quasiplatonic surfaces. We know that a generating triple (a, b, c) of a group G defines a quasiplatonic action on some surface S. It is well known, e.g., [81] that S has a symmetry ϕ normalizing the action of G if and only if there is an involutary automorphism θ satisfying one of the following conditions:
(1) $\theta(a) = a^{-1}, \theta(b) = b^{-1}$,
(2) $\theta(a) = b^{-1}, \theta(b) = a^{-1}$.

The automorphism θ is defined by $\theta(g) = \phi g \phi^{-1}$.

Some examples of families of quasiplatonic actions with symmetries are given in [81], [22], [21], and [84]. For G an abelian group, it is straightforward to see that the surface is symmetric [81]. For $G = \mathrm{PSL}(2, q)$ it can be shown that every triple yields a symmetric surface [81] and [84]. These examples lead to the following problems.

Problem 64. *Are there any conditions on a group G such that the quasiplatonic actions of G are always symmetric?*

Problem 65. *More specifically, for what simple groups are quasiplatonic surfaces always symmetric?*

More generally, we have:

Problem 66. *For a given group G there are only finitely many generating triples. What proportion of the triples yield symmetric actions? Of particular interest is the case where the fraction is small.*

7.2. Existence of symmetries. If there is a symmetry ϕ normalizing the action of G, we say that the action is a "symmetric G-action". There is a subtlety here, namely a surface might be symmetric but the symmetry does not normalize the G-action. In this case $\epsilon(G)$ is strictly contained in $\mathrm{Aut}(S)$.

We can build a surface with G-action from a quotient surface pair (T, B_G) and a generating vector $\mathcal{V}$. We would like to do the same for symmetries. Assuming the existence of ϕ, there is a quotient symmetry $\overline{\phi}$ of T. To lift a symmetry $\overline{\phi}$ of T to S two conditions that must be considered are:

- The branch points in B_G and the branch orders themselves must be invariant under $\overline{\phi}$.
- The hypothesised automorphism of G, $\theta(g) = \phi g \phi^{-1}$ imposes some constraints on the generating vector.

For example the conditions are quite simple in the quasiplatonic section as discussed above. For greater numbers of branch points the problem becomes more complex. The following problem is interesting though it might be easy.

Problem 67. *For low orbit genus h and low number of branch points t, assume we are given the orbit space (T, B_G) and a generating vector $\mathcal{V}$ for a G-action.*

- *Enumerate the possible geometric configurations of B_G that would be invariant under some symmetry $\overline{\phi}$ of T.*
- *For each geometric configuration above determine the constraints that the hypothesised automorphism of G, $\theta(g) = \phi g \phi^{-1}$ imposes upon the generating vector.*

7.3. Symmetric n-gonal actions. Assume that we have solved Problem 67. Here are some problems for general n-gonal surfaces that are similar to the quasiplatonic case.

Problem 68. *For a specific (non-abelian) group G, work out all the actions with four branch points and determine which ones are symmetric.*

Problem 69. *Are there any (non-abelian) groups G for which all the actions with four branch points and correct geometry of B_G are symmetric?*

Problem 70. *Repeat the same problems as above with more than 4 branch points.*

Problem 71. *Repeat the same problems as above with orbit genus 1 and one or more branch points.*

Since, for $t > 3$, the geometric constraints show that symmetric surfaces form a proper, possibly non-empty, subset of all surfaces with G-action of a given topological type, this prompts the following problem.

Problem 72. *For a given group action G with a non-empty equisymmetric stratum, determine the structure of the symmetric surfaces as a subset of the equisymmetric stratum. Start with 4 branch points.*

8. Algorithms, computations, and explicit methods

The advancement of computational power over the last 30 years has added rich new areas of study, and within each of the areas discussed above there are questions which may be answered (or partially answered) by computational methods. Computational data has also led to new conjectures, as well as evidence for (and against) other conjectures. For many of the problems given in the other sections, such as Problems 6, 7, 8, 14, 46, 56, 67, and 68, computation is an integral part of a strong solution, either for exploration or complete classification. For almost all problems in this paper, computation can be used to generate examples beyond toy examples.

8.1. Classifications. Breuer [13] began the new millennium with a complete classification of automorphisms of Riemann surfaces up to genus 48. His contribution was to devise an algorithm to generate a list of all groups and corresponding signatures for which there is a surface kernel epimorphism $\eta : \Gamma \to G$ for a fixed genus. His computer code outputs group and signature pairs. While his algorithm did compute generating vectors, he did not record these generating vectors in his original data.

The only obstacle to computing higher genus data is that his algorithm requires a complete classification of groups of order up to the maximum order for the given genus. The computer algebra programs Magma [9] and GAP [35] have complete classifications of groups of order up to 2000 (except for order 1024). Some of Breuer's data (along with additional information about the corresponding Riemann surfaces) may be found at [59].

In addition, Conder [32] computed all automorphism groups of size greater than $4(g-1)$ for a given genus g, up to genus 101, by using the theory in Section 2.1.2 and the *LowIndexNormalSubgroups* routine in Magma. The size conditions on the automorphism group guarantees that $g_0 = 0$ and that the cover is branched at 3 or 4 places. His data includes the generating vectors for each action up to simultaneous conjugation, i.e., up to the action of an inner automorphism of the automorphism group.

While Breuer and Conder both provide extensive lists of group and signature pairs, their algorithms do not compute equations (or models) of the corresponding Riemann surfaces. Recall the definition of Hurwitz spaces from Section 4.1. In particular, if we are given a group G and an unordered r-tuple $\mathbf{C} = (C_1, \ldots, C_r)$ of conjugacy classes of G, the Hurwitz space of this data consists of those surfaces with ramification type $\mathbf{C}$. There are many cases where computing $\mathbf{C}$ is feasible, but where we do not know equations for the family of curves in the Hurwitz space. Yet there are many applications where knowing equations for the curves would give additional information.

PROBLEM 73. *Is there an explicit algorithm that, given as input the ramification type $\mathbf{C}$ of the Hurwitz space, produces as output explicit equations for the associated Hurwitz space in a reasonable amount of time?*

Gröbner basis methods may play a role in these computations, and their run time is double exponential in the input size, so one interpretation of "reasonable amount of time" is doubly exponential. Even the special case of producing equations for Riemann surfaces "with many automorphisms" is already interesting. Swinarski has some work in this direction [83].

Recall Belyi maps which were discussed in Section 6.1. There are many computational questions surrounding these objects, and which have been already catalogued by Sijsling and Voight in [78] and which we will not reproduce here, except to highlight one particular question.

PROBLEM 74. *Suppose we are given a Beyli map $\beta : S \to \mathbb{P}^1$ with coefficients in a number field K, and a fixed complex embedding $K \hookrightarrow \mathbb{C}$. Can we compute the permutation triple which describes the monodromy around the three branch points 0, 1, and ∞?*

While there are some numerical methods which can be used to compute monodromies (see [78, Section 8] for more details), guaranteeing a rigorous output with those numerical techniques may not be tractable. Are there other techniques which might work here, particularly given that there are already known techniques to find the monodromy group itself? And, as with any computational problem, run time estimates for such algorithms are always welcome!

8.2. Equivalence relations. While Breuer's data computes generating vectors up to simultaneous conjugation, as discussed in Section 2.2, it is often sufficient to only ask for classifications of the actions up to braid, topological or conformal equivalence. For the next few problems, we need to reformulate the definition of topological equivalence for surface kernel epimorphisms. Two actions η_1 and η_2 defined as in Section 2.2 are *topologically equivalent* if there exists an $\omega \in \mathrm{Aut}(G)$ and $\phi \in \mathrm{Aut}^+(\Gamma)$ (the orientation preserving automorphisms of the Fuchsian group Γ) so that the following diagram commutes [15].

$$\begin{array}{ccc} \Gamma & \xrightarrow{\eta_1} & G \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\omega} \\ \Gamma & \xrightarrow{\eta_2} & G \end{array}$$

Notice this means that $\eta_2 = \omega \circ \eta_1 \circ \phi^{-1}$ where ϕ is an element of $\mathrm{Aut}^+(\Gamma)$ and $\omega \in \mathrm{Aut}(G)$. As such, two actions are topologically equivalent precisely when they are in the same orbit under the action of $\mathrm{Aut}(G) \times \mathrm{Aut}^+(\Gamma)$ [15, Proposition 2.2].

This translates the definition of topological equivalence to an algebraic condition and using this result for cases where $h = 0$, code is implemented in Sage [3] which computes, given a group and a signature, the orbits under the corresponding action and returns one representative of each orbit. There is also similar code in GAP to compute mapping class orbits [50] including the cases where $h > 0$. These programs work well for many examples of modest genus or with a modest number of branch points. However, these programs are not effective as the size of the group or the number of branch points get large. In addition, little has been done computationally to determine actions up to conformal equivalence. As such, the following two problems are of interest.

Problem 75. *Improve on current algorithms to compute orbits under topological equivalence.*

Problem 76. *Determine algorithms to compute orbits under conformal equivalence, given a group and signature pair, or a group and ramification type.*

8.3. Problems on enumerating actions. Enumerating generating vectors leads to our last set of computational problems. Consider the following two action spaces. Given a G-signature, or ramification type, $(C_1, \ldots, C_t)$ and any subgroup $H \leq G$, define the sets

$$X_H(C_1, \ldots, C_t) = \left\{ \begin{array}{c} (x_1, \ldots, x_t) \in (C_1 \times \cdots \times C_t) : \\ x_1 \cdot \cdots \cdot x_t = 1, \langle x_1, \ldots, x_t \rangle \leq H \end{array} \right\} \text{ and}$$

$$X_H^\circ(C_1, \ldots, C_t) = \left\{ \begin{array}{c} (x_1, \ldots, x_t) \in (C_1 \times \cdots \times C_t) : \\ x_1 \cdot \cdots \cdot x_t = 1, \langle x_1, \ldots, x_t \rangle = H \end{array} \right\}.$$

The special case $X_G^\circ(C_1, \ldots, C_t)$ is the set of generating vectors. We may use Hall Möbius inversion to compute the values $\{|X_H^\circ(C_1, \ldots, C_t)|, H \leq G\}$ from the set of values $\{|X_H(C_1,\dots,C_t)|, H \leq G\}$ There are formulas using character theory [51] that can be used to compute $X_H(C_1,\dots,C_t)$. To use the formulas we also need to know how $H \cap C_j$ breaks up into conjugacy classes of H, and computation of the character tables of H.

These formulas lead to the following questions.

Problem 77. *Find an efficient algorithm that, given a potential G-signature $(C_1,\dots,C_t)$, determines the number of generating vectors in $(C_1 \times \cdots \times C_t)$. Obviously, we only need to look at subgroups such that $H \cap C_j$ is non-empty for every C_j.*

Problem 78. *Given a potential G-signature $(C_1,\dots,C_t)$ determine how many braid equivalence classes are contained in $(C_1 \times \cdots \times C_t)$*

As with many other problems in the paper, solving these problems for the n-gonal case would be a good starting point.

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