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On the Capacity of Network Coding for Random Networks

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Abstract

We study the maximum flow possible between a single-source and multiple terminals in a random graph (modeling a wired network) and a random geometric graph (modeling an *ad-hoc* wireless network). We show for both models that the maximum flow is tightly concentrated around its mean. Specifically for n nodes in a network such that any two nodes have a probability p of connectivity, the maximum flow between a single source and multiple terminals is approximately $(n-1)p$ for large n. Similar results are obtained for the case of n randomly located nodes on the unit square.

1 Introduction

Consider a communication network where one source node wants to transmit information through a network to multiple terminal nodes. This paper considers the problem of finding the capacity of this scenario under the assumption of random node placement. The capacity under consideration here is the graph-theoretic max-flow capacity, not the capacity in the information-theoretic sense. It is a known fact that routing achieves the max-flow minimum cut [1] capacity of a network when transmissions are point to point (for a wired network). However, recent results have shown that for the single source - multiple terminal case, the information rate to each terminal is the minimum of the individual max-flow bounds over all source-terminal pairs under consideration and that in general we need to code over the links in the network to achieve this capacity [2]. Li et al. [3] have shown that linear network coding is enough to achieve the capacity of the transmission of a single source to multiple terminals.

It is important to clearly differentiate between routing and network coding. We say that a network employs routing when each node in the network performs only a "replicate and forward" strategy. Thus each node can create multiple copies of a received packet and forward it on different lines. "Network coding", on the other hand, refers to the situation when each node has the ability to perform operations such as linear combinations on the received data and then send the result on different lines. So, routing is a special case of network coding.

The usefulness of network coding can be understood by considering a simple topology shown in Fig. 1, which we borrowed from [4]. We assume that each link can transmit a single-bit, error-free and delay-free. Observe that performing network coding (as shown

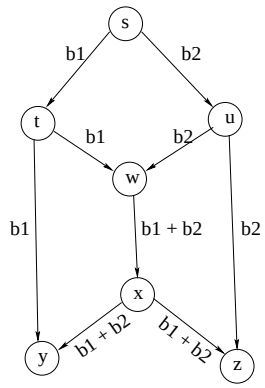


Figure 1: Network with source s and terminals y and z

in Fig. 1) enables transmission of both b_1 and b_2 to both the terminals y and z simultaneously whereas routing would require two transmissions. In this paper only the source and the terminal nodes are communicating with each other and the rest of the nodes are acting as relays. In the subsequent sections we shall use the term "s-t minimum cut" for the source-terminal minimum cut and the term "global minimum cut" for the minimum of the s-t minimum cuts over all possible source-terminal pairs in the network.

Sections II and III prove concentration theorems for random graphs [5] (a model for wired networks) and random geometric graphs [6] (a model for wireless networks) respectively. In both cases the s-t minimum cut is concentrated about its mean. Section IV provides simulations that confirm the concentration results and reveal the mean to be approximately $(n-1)p$ where p is the probability of an edge. Using [2] this implies that for large n rates close to np can be reliably sent from a single source to multiple terminals via linear network coding [3] for wired networks. For wireless networks however the result is an upper bound on the achievable information transfer [7] since effects such as interference cannot be modeled.

2 Wired Networks - The Random Graph Model

We consider the random graph as a model of a wired network. This is not the only model for wired networks in general. The Internet, for example, has been shown to obey a power-law [8]. However the random graph gives us a good framework to analyze a network whose topology is unknown. The random graph $G(n,p)$ is defined in the following manner :

Definition 1 (Random Graph) *Given $n \in \mathbb{N}$ and $0 \leq p \leq 1$, we define $G(n,p)$ as a graph on the vertex set $V = 1, ..., n$, generated by choosing an edge between any two vertices with probability p.*

The number of edges in the random graph is a random variable. The minimum cut between any source-terminal pairs in the graph is also a random variable. The mean of the s-t minimum cut is equal to the maximum information flow the pair can achieve on average. We now provide bounds on the tail distribution of s-t minimum cuts in the random graph defined above.

In his derivation of fast randomized algorithms for finding the minimum cut of a deterministic graph, Karger [9] proved a concentration result for random subgraphs or

"skeletons" of the original deterministic graph with global minimum cut c and s-t minimum cut v (for some s, t). For the theorem below, a skeleton G_p of the original graph G is constructed by retaining each edge of G with probability p.

Theorem 1 *(Karger's Concentration Result) Let G be any graph with global minimum cut c and let,*

$$p = \frac{3(d+2)(\ln n)}{\epsilon^2 c} \tag{1}$$

where $d \in \mathbb{Z}^+, \epsilon \in \mathbb{R}^+$ such that $p \leq 1$. Suppose the s-t minimum cut of G (for some s, t) has a value v. Then with probability $1 - O(1/n^d)$ the s-t minimum cut in G_p has value between $(1-\epsilon)pv$ and $(1+\epsilon)pv$, and the global minimum cut has value between $(1-\epsilon)pc$ and $(1+\epsilon)pc$.

Note that in the above theorem,

$$\epsilon = \sqrt{\frac{3(d+2)(\ln n)}{pc}} \tag{2}$$

As we shall see below for the graphs of interest, $c \propto n$ and consequently $\epsilon = \mathrm{O}\left(\sqrt{\frac{\ln n}{n}}\right)$ i.e. ϵ is small for large n. We use Karger's theorem to prove the following corollary.

Corollary 1 *(Concentration Result for Complete Graphs) For a random graph $G(n,p)$ for $\epsilon = \sqrt{3(d+2)(\ln n)/p(n-1)}$, the probability that the value of some s-t minimum cut is more than $(1+\epsilon)p(n-1)$ or less than $(1-\epsilon)p(n-1)$ is $O(1/n^d)$. Also the probability that the global minimum cut is more than $(1+\epsilon)p(n-1)$ or less than $(1-\epsilon)p(n-1)$ is $O(1/n^d)$.*

Proof: Let $\tilde{G}$ be an instance of $G(n,p)$ and let G_c denote the complete graph (every vertex connected to every other vertex) on n vertices. In Theorem 1 we choose G to be G_c.

Any s-t minimum cut in G_c has a value $(n-1)$ since every node is connected to every other node. Since the global minimum cut is the minimum value of a cut over all source-terminal pairs, the global minimum cut is $(n-1)$ as well. Now using (2) we can compute a value of ϵ for a corresponding value of d. If $\epsilon \leq 1$ then by Theorem 1 the s-t minimum cut and the global minimum cut of $\tilde{G}$ have a value between $(1-\epsilon)p(n-1)$ and $(1+\epsilon)p(n-1)$ with high probability for large n. ■

The higher d is in (1) and (2), the lower the tail probability. At the same time, a higher d causes ϵ to increase. There is a trade-off between these two parameters that decides the tightness of the bound.

Thus, in a random graph there is a strong case for using network coding since the s-t minimum cut provably remains more or less concentrated about its mean. On average we won't lose much because of the random nature of the graph. Note that for a wired network the capacity of the single-source multiple-terminal information transfer (i.e. the minimum of all the s-t minimum cuts) is actually achievable. There exists a network code that can be found in polynomial time [10] that achieves this capacity. However the result above is an "existence result", we do not provide an algorithm for finding the network code.

3 Ad Hoc Wireless Networks - The Random Geometric Graph Model

At first one might consider network coding inappropriate for a distributed wireless network because transmissions from relatively simple distributed wireless nodes (such as wireless sensor networks) are typically omni-directional, precluding the transmission of different bits from the same node to different links at the same instant of time and in the same frequency band. However communication has been shown to dominate all other sources of energy consumption in a sensor network. So, in order to save power, wireless sensor nodes typically will go into a sleep mode [11] from which they periodically awaken to listen for transmissions. Furthermore, nodes negotiate time slots and frequency slots with which to communicate for both transmission and reception, also with a desire to minimize power drain. Under these practical assumptions network coding ideas would be possible to implement in a wireless network. Observe that many sensor networks would need a sensor node to periodically send data to a set of other nodes. Network coding might provide a viable solution to the low-energy single-source multiple terminal information transfer problem where distinct edges correspond to different frequencies or time slots in a single transmission epoch.

The random graph model of Section II is definitely not a realistic model for a wireless ad-hoc network or sensor network because it places edges between nodes independent of the distance between them. Distance is a critical factor in determining the connectivity properties of a wireless network since propagation losses cause the power of the signal to fall off as $r^{-\alpha}$ where $2 \leq \alpha \leq 4$. To properly model wireless networks, consider a class of graphs known in mathematical literature as random geometric graphs [6].

Definition 2 *Random geometric graphs $G_n(d,r), 0 \leq r \leq 1$ are defined on the unit cube $[0,1]^d, d \geq 2$ by randomly choosing a sequence $I = [I_1, I_2..., I_n]$ of independent and uniformly distributed points (vertices) on $[0,1]^d$ and by choosing a distance metric $d(i,j)$ that gives the distance between the vertices $i, j \in V$. If $d(i,j) \leq r$ then there exists an edge between i and j.*

In general we could assign a weight to the edge. However for simplicity we shall assign unit weight to all edges. In this work we consider the L^2 norm (standard Euclidean distance) as our distance metric. Compared to the random graph, the random geometric graph is an inherently harder model to work with since the nature of the graph induces dependencies between edges as shown below. In the random graph, the edge probabilities are independent and identically distributed. However, as we shall see, for the random geometric graph the edge probabilities are identically distributed but not independent.

Consider three vertices i, j, k in the random geometric graph as illustrated in Fig. 2. The region R_i is the circle centered at i. We denote the fact that i and j are connected by $i \to j$ (Fig. 2). Since node placements are i.i.d., it follows that

$$\begin{aligned} P[i \to k] &= P[i \to j] \\ &= \frac{\text{Area}(R_i)}{\text{Area}(U)} \end{aligned} \tag{3}$$

However, the critical point is that,

$$P[i \to k | i \to j, k \to j] \neq P[i \to k] \tag{4}$$

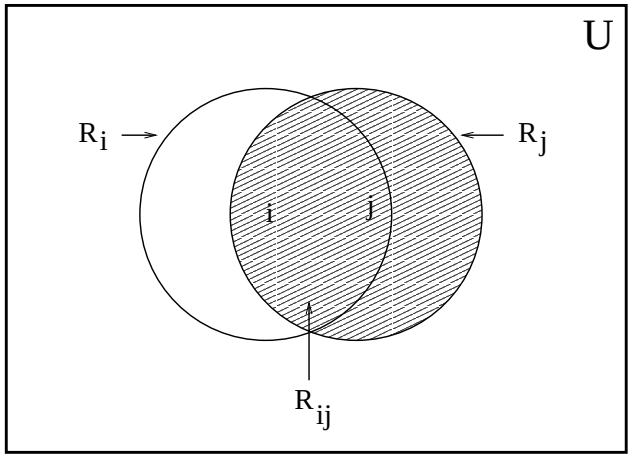


Figure 2: Note that if a third node k is connected to j then it surely falls in the shaded area R_j. If it falls in $R_{ij} = R_i \cap R_j$ then it is also connected to i.

This is because, from Fig. 2

$$P[i \to k | i \to j, k \to j] = \frac{\text{Area}(R_{ij})}{\text{Area}(R_j)} \neq \frac{\text{Area}(R_i)}{\text{Area}(U)} \tag{5}$$

Note that in the random graph case,

$$P[i \to j | j \to k, i \to k] = P[i \to j] \tag{6}$$

because edges are chosen independently. The analysis in [9] proceeds via the Chernoff bound that is suited to the random graph model since the edge probabilities in that case are independent. However we cannot use that analysis for the random geometric graph since the edge probabilities are surely not independent based on the discussion above.

In this section we analyze the behavior of the s-t minimum cut and the global minimum cut in a random geometric graph. This analysis only provides an upper bound on the amount of information flow possible since max-flow bounds are upper bounds in general for wireless systems [7]. Let,

$$X_{i,k} = \begin{cases} 1 & \text{if } i \to k, \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Without loss of generality consider a cut in which the first k nodes, labeled $0, 1, ..., (k-1)$ are in one set and the $(n-k)$ nodes, labeled $k, k+1, ..., (n-1)$ are in the other. We define a function $f(X_{0,k}, X_{0,k+1}, ..., X_{0,n-1}, ..., X_{k-1,k}, X_{k-1,k+1}, ..., X_{k-1,n-1})$ (henceforth just f for convenience) that finds the value of the cut.

$$f(X_{0,k}, X_{0,k+1}, ..., X_{0,n-1}, ..., X_{k-1,k}, X_{k-1,k+1}, ..., X_{k-1,n-1}) = \sum_{i=0}^{k-1} \sum_{j=k}^{n-1} X_{i,j} \tag{8}$$

Let

$$N_i = E[f | \underbrace{X_{0,k}, X_{0,k+1} ...}_{i \text{ terms}}] \tag{9}$$

Thus, N_i is the expected value of the cut given that we know the status of the first i links out of the possible $k(n-k)$ links. We can show that N_i is a martingale [12]. Also we have,

$$|N_i - N_{i-1}| \leq 1 \tag{10}$$

This follows directly from the fact that the cut can change by at most 1 through the disclosure of the status of a link.

Note that $N_0 = E[f]$ and $N_{k(n-k)} = f$. Using Azuma's inequality [12], we have the following theorem.

Theorem 2 *Let C_k be a cut in a random geometric graph $G(V, E)$ that is defined by partitioning the vertex set (V) into a set V_k $(|V_k| = k)$ and the complementary set $\bar{V}_k$ $(|\bar{V}_k| = n - k)$. Then,*

$$P(|C_k - k(n-k)\mu| \geq \epsilon k(n-k)\mu) \leq 2e^{-\frac{\epsilon^2 k(n-k)\mu^2}{2}} \tag{11}$$

where $\mu = P[i \to k] = E[X_{i,k}]$ is the probability that two nodes i and k are connected.

Proof: The proof follows by choosing the sequence $N_1, N_2...$ as our martingale sequence. By (10) we satisfy the bounded difference requirement. Applying Azuma's inequality we obtain,

$$P(|N_{k(n-k)} - N_0| > t) \leq 2e^{-\frac{t^2}{2k(n-k)}} \tag{12}$$

Choosing $t = \epsilon k(n-k)\mu$ we have the result. ■

Now we show that as in the case of the random graph, all cuts in a random geometric graph are concentrated around their mean value.

Theorem 3 *Let $\epsilon = \sqrt{\frac{4(\gamma+1)\ln n}{n\mu^2}}$, with μ as defined above. If $\epsilon \leq 1$, then with probability $1 - O(1/n^\gamma)$, every cut in a random geometric graph G has a value between $1 - \epsilon$ and $1 + \epsilon$ times its expected value.*

Proof: Let $m = 2^n - 2$ be the number of cuts in the graph G.

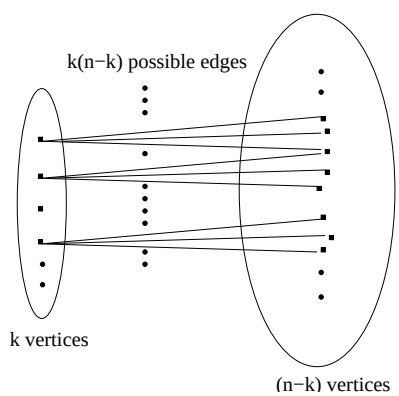


Figure 3: There are $\binom{n}{k}$ cuts of expected size $k(n-k)\mu$. One such cut is illustrated here.

There are $\binom{n}{k}$ cuts that divide the graph into two sets with k and $n-k$ vertices respectively as shown in Fig. 3. We call them, cuts of type $\mathscr{C}_k$. Let p_k = probability that a cut of type $\mathscr{C}_k$ deviates by more than ϵ from its expected value, in a random instance of G. Then by the union bound,

$$P(\text{Some cut deviates by more than } \epsilon) \leq \sum \binom{n}{k} p_k \tag{13}$$

From Theorem 2 we obtain

$$p_k \leq 2e^{\frac{-\epsilon^2 k(n-k)\mu^2}{2}} \tag{14}$$

For notational convenience we define $\beta = e^{-\frac{\epsilon^2 n\mu^2}{2}}$. Then we can upper bound the RHS of (13) as,

$$\begin{aligned}
\sum \binom{n}{k} p_k &\leq 2\sum_{k=1}^{n-1} \binom{n}{k} \beta^{n(\frac{k}{n})(1-\frac{k}{n})} \\
&\overset{(a)}{=} 2[\sum_{k=1}^{n/2} \binom{n}{k} \beta^{n(\frac{k}{n})(1-\frac{k}{n})} + \sum_{k=n/2+1}^{n-1} \binom{n}{k} \beta^{n(\frac{k}{n})(1-\frac{k}{n})}] \\
&\overset{(b)}{\leq} 2[\sum_{k=1}^{n/2} \binom{n}{k} \beta^{k/2} + \sum_{k=n/2+1}^{n-1} \binom{n}{k} \beta^{n(\frac{1}{2}-\frac{k}{2n})}] \\
&\overset{(c)}{\leq} 4[(1+\sqrt{\beta})^n - (1+(\sqrt{\beta})^n)] \\
&= 4[(1+e^{-\frac{\epsilon^2\mu^2 n}{4}})^n - (1+e^{-\frac{\epsilon^2\mu^2 n^2}{4}})]
\end{aligned} \tag{15}$$

where inequality (b) follows from the fact that $(\frac{k}{n})(1-\frac{k}{n}) > \frac{k}{2n}$, for $\frac{k}{n} \in [0, 1/2]$ and $(\frac{k}{n})(1-\frac{k}{n}) > (\frac{1}{2} - \frac{k}{2n})$ for $\frac{k}{n} \in [1/2, 1]$. Inequality (c) follows from the fact that $\sum_{k=1}^{n/2} \binom{n}{k} \beta^{k/2} \leq \sum_{k=1}^{n} \binom{n}{k} \beta^{k/2}$. Now if we choose, $\epsilon = \sqrt{\frac{4c\ln n}{n\mu^2}}$, where $c > 2$ we can simplify the above expression (for large n) as,

$$4\Big[\Big(1+\frac{1}{n^c}\Big)^n - \Big(1+\Big(\frac{1}{n^c}\Big)^{n^2}\Big)\Big] \approx O(1/n^\gamma) \quad \text{where } \gamma = c-1 \tag{16}$$

There is a trade-off between μ, ϵ and γ that decides the tightness of the bound. At a given μ, a higher γ gives a lower tail probability, but at a higher ϵ. Note that a high value of μ makes it easier to achieve a lower value of ϵ.

Again for an *ad-hoc* network modeled by a random geometric graph we do not lose much because of a lack of control over placement of the nodes as long as we have a sufficiently high number of nodes.

4 Simulations and Discussion

4.1 Boundary effects in random geometric graphs

We have derived the concentration results for both random graphs and random geometric graphs. Since all the s-t pairs are statistically the same in a random graph, the mean of the s-t minimum cut can be calculated by averaging over randomly picked s-t pairs. However this approach does not work for a random geometric graph where nodes at the boundary usually have fewer links than those at the center. The s-t pairs at the boundary will have smaller minimum cuts. According to Theorem 3 each cut is still concentrated around its own mean, but the average over different s-t minimum cuts will become smaller. Also due to the boundary effects, the global minimum cut will be dominated by the s-t minimum cuts at the boundary and will consequently achieve a lower value. The same boundary effects are also reported in [13].

One way to avoid the boundary effects is to use a toroidal distance metric [13]. With a toroid, nodes at one boundary of a hypercube are considered to be close to the nodes at the opposite boundary i.e. nodes at the left boundary of a square can have links with nodes at the right boundary, and nodes near the top of the square can have links with

those at the bottom. All the s-t minimum cuts are equivalent in the toroidal case. Fig.

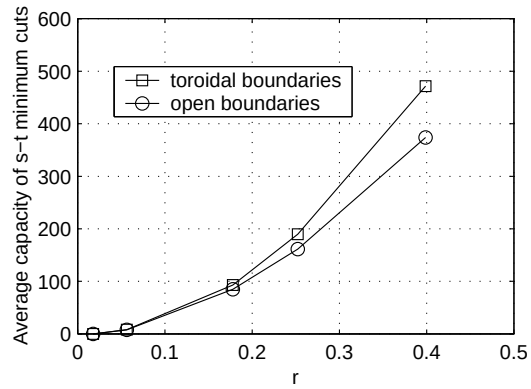


Figure 4: Boundary Effects in Random Geometric Graphs

4 shows the boundary effects in random geometric graphs with 1000 nodes varying with respect to r, where r is the access range of a node. The gap between the curves for the toroidal and open boundary case gets bigger as r increases. This is because the number of nodes residing in the boundary increases with r.

4.2 Mean of s-t minimum cuts

For a random geometric graph with n nodes which are uniformly distributed in the two dimensional box $[0,1]^2$, a pair of nodes are connected if and only if the distance between them is less than r. Denote $p_d(x)$ as the probability density function of the squared distance between two arbitrary nodes for the open boundary case, where $x = r^2$. Then,

$$p_d(x) = \begin{cases} \pi - 4\sqrt{x} + x & 0 \le x < 1 \\ 2\arcsin(\frac{2}{x} - 1) + 4\sqrt{x-1} - x - 2 & 1 \le x < 2. \end{cases} \tag{17}$$

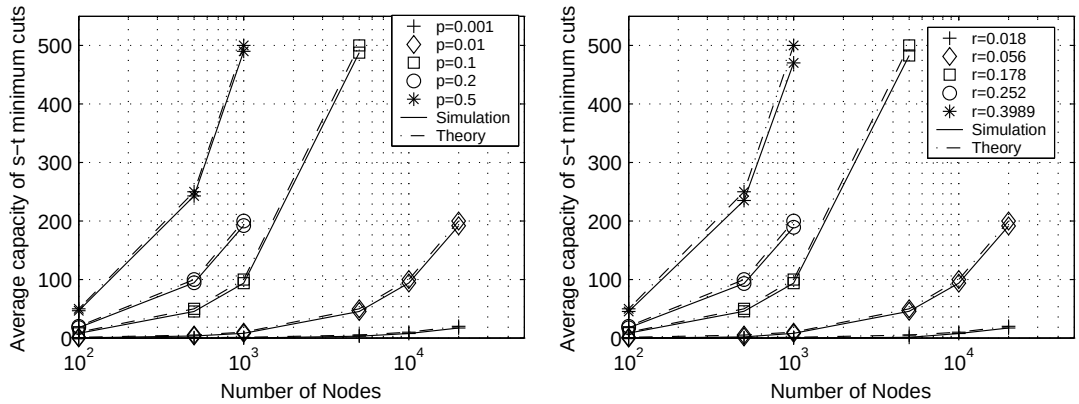
In the toroidal case, the squared distance between two nodes has the density function,

$$p'_d(x) = \begin{cases} \pi & 0 \le x < 1/4 \\ 2\arcsin(\frac{1}{2x} - 1) & 1/4 \le x < 1/2. \end{cases} \tag{18}$$

Simulation shows that as far as the s-t and global minimum cuts are concerned, a random graph $G(n,p)$ and a random geometric graph with parameter's n and r under the toroidal distance metric are equivalent if p and r are related by the cumulative distribution function of (18). In particular, the means of the s-t minimum cuts of both the random graph and the random geometric graph defined on a torus are approximately $(n-1)p \approx np$, where n is the number of nodes, p is the probability of presence of an edge for the random graph and the corresponding probability $p'_d(X \le r^2)$ computed according to (18) for the random geometric graph. The means of s-t minimum cuts for different random graphs and random geometric graphs along with the corresponding np values (labeled "Theory") are presented in Fig. 5(a) and Fig. 5(b) respectively.

4.3 Concentration of minimum cuts

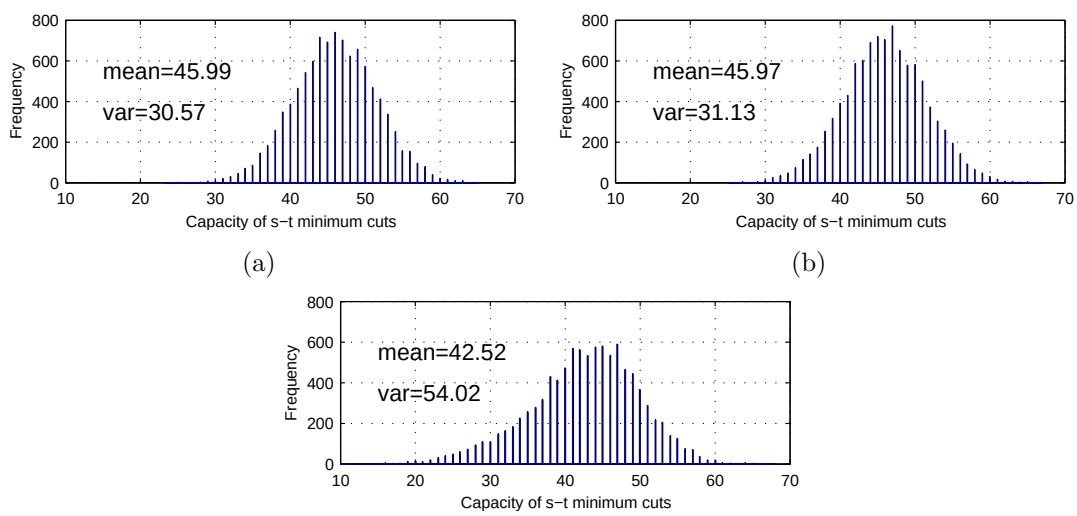
Theoretically the tail bounds for the random geometric graph are weaker than those for the random graph because the former are obtained via martingale arguments while the latter are obtained via Chernoff bounds. Simulation results on random graphs with



(a) (b)

Figure 5: (a) Random Graphs: s-t minimum cuts and their means (b) Random Geometric Graphs: s-t minimum cuts and their means

$n = 1000$, $p = 0.05$ and random geometric graphs with same number of nodes and $r = 0.1262$ (chosen such that $p'_d(X \leq r^2) = p$) are presented in Fig. 6. The resemblance of Fig. 6(a) and Fig. 6(b) suggests that the difference might be negligible in the toroidal case. However Fig. 6(c) shows that in the open boundary case the histogram spreads to the lower end, which corresponds to those s-t minimum cuts comprised of nodes at the boundary. It is worth noting that for small r, the boundary effects are negligible as is also shown in Fig. 4.



(a) (b)

(c)

Figure 6: Histograms of s-t minimum cuts for (a) random graphs, (b) random geometric graphs with toroidal boundaries, (c) random geometric graphs with open boundaries. Note the similarity between (a) and (b) and the difference from (c).

5 Conclusion

We presented basic results on the concentration of the s-t minimum cut and global minimum cut for a random graph and a random geometric graph. It is interesting

to note that even though the random geometric graph has dependencies among edges, the s-t minimum cut still behaves asymptotically as that of a random graph with an appropriately chosen probability based on (17) or (18). Steps have been taken by Koetter and Medard [4] towards the construction of network codes that achieve the promised capacity. Thus, as far as single-source multiple-terminal information transfer in wired networks goes network coding seems to be a strong competitor to routing. Our results on random geometric graphs also suggest that it might be a viable alternative for wireless networks.

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